Non-Standard Analysis

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Abstract

Mathematics contains naturally occurring continuous phenomena, such as the geometric line. One project in mathematics is to provide a characterisation of the continuous number system, the *arithmetic continuum*. The standard account says that this is uniquely captured by $\mathbb{R}$.

I discuss an alternative, $\mathbf{R}$, which enriches $\mathbb{R}$ by including infinitely small and (some) infinite numbers. I outline some reasons to think that $\mathbf{R}$ better captures certain continuous phenomena, fits well with student intuitions about analysis, and has mathematical applications for which there is no standard interpretation. I consider a criticism of $\mathbb{R}$, that these supporting argument superficially seem to prioritise $\mathbb{R}$. This may be explained by viewing $\mathbb{R}$ as a well-behaved subsystem. This gives reason to think that $\mathbf{R}$ may be a better candidate that $\mathbb{R}$ at capturing the intuitive notion of continuity.

Crucial to continuity is capturing the intuitive notion of 'gap-free'. Accounts with favour $\mathbb{R}$ rely on a particular notion of gap, Dedekind-gap, which, amongst other problems, excludes otherwise gap-free structures, may be circular, and is closely tied up with the background set theory. An alternative notion, 'elementary gap', favours $\mathbf{R}$ as a candidate continuum. This notion of 'gap-free' weathers these criticisms, though somewhat questionably requires a fixed language for analysis. So, $\mathbb{R}$'s claim to the continuous throne is not uniquely strong.

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Contents

1 Tiny Numbers

The foundations of mathematics often addresses the discrete and the very large, like Cantor's infinite cardinals, or the discrete steps of iterative set theory. But there are also rich questions about the small and the continuous. I focus on one account of these, Robinson's non-standard analysis (NSA), with its extension of $\mathbb{R}$, the 'hyperreals'. My principal philosophical aims are:

1. To provide reason to take NSA more seriously.
2. To show that the standard conception ($\mathbb{R}$) is not the unique candidate for continuity and coordinatisation of the geometric line: the hyperreals are also plausible.[1]

There is a good case to be made that the hyperreals are more than well-founded fictions, Hilbertian 'ideal' mathematics, or methodological tricks. Their interest is not limited to the model theorist, as a pseudo-numerical object. Robinson once remarked "Number systems, like hair styles, go in and out of fashion" [55]. But there are reasons to think the hyperreals are better justified than other hair styles.

Robinson's hyperreals field includes infinitesimals:

Definition 1 (Infinitesimal [5]:77)**.** *An* infinitesimal *is a quantity η such that $0 < |\eta| < \frac{1}{n}$ for every natural number $n > 0$*

There are several infinitesimal analyses. Many are radical. Conway's surreal numbers (maximally) generalise the idea of Dedekind cuts [44]. Their field is class-sized, and is more or less the broadest possible notion of 'number' ([51]:§6). Meanwhile, smooth infinitesimal analysis, inspired by category theory, violates LEM, so has a non-classical logic ([3]:5).

Non-standardness also occurs more generally in mathematics (thanks e.g. to the Löwenheim-Skolem Theorems). Hence, Nelson's Internal Set Theory builds non-standardness into the foundations. It enriches the syntax of the background set theory with a predicate 'is standard', and stipulates additional axioms governing the existence of non-standard elements,

[1] I make no claim about the nature of mathematical concepts or objects (see [5]:146). Instead, *whatever* position the real numbers occupy, as 'the' description of continuity, could be occupied by the hyperreals.

such as infinitesimals [50]. But some claim that unintended interpretations of set theory are infelicitous to the meaning of our words ([48]:133).

However, my focus, NSA *does not* seem to be infelicitous, due to the transfer principle. This roughly says that NSA doesn't change the nature of $\mathbb{R}$ as a subsystem. Instead, it enriches (or reveals the richness) of words concerning continuity.

I begin by sketching the construction of the hyperreals. In §3, I outline the mathematical and pedagogical motivations for NSA. Pedagogically, limits and the ε-δ method are hard. Meanwhile, non-standard methods in calculus and analysis resemble those of basic arithmetic, facilitating education in these areas. NSA also seems better aligned with certain intuitions, and is perhaps more explanatory. This raises the issue of the apparent primacy of the reals. I argue that this is not definitive. Outright mathematical advantages of NSA include simplifications of definitions and proofs ([19]:vii), and mathematical results which are only known using non-standard methods. Moreover, the hyperreals seemingly have a rich higher-order theory.

Next, I outline the connection between continuity in gaps. In §5, I reconstruct arguments for the common belief that the reals are the obvious candidate explanation of continuity and the geometric line. I critique these in §6, arguing that Dedekind gaps may *not* be the relevant notion of gaps for continuity. Elementary gaps are suggested as an alternative in §7, concluding that NSA is also a strong candidate for capturing the intuitive notion of continuity.

2 Hyperreals

Let $\mathbb{R}$ be the (ordinary) real numbers as a dense linear order, with suitable constants and functions. This system is expanded to the hyperreals, just as $\mathbb{R}$ expands $\mathbb{Q}$. There are several ways to construct these non-standard reals, each requiring the compactness theorem or some version of an ultrapower construction ([42]:208).[2] Robinson's own type-theoretic construction used compactness ([54]:19). Some approaches to NSA change the underlying set theory, notably Nelson's [50], but NSA can be implemented

[2]The ultrapower construction quotients $\mathbb{R}^{\mathbb{N}}$ by a free ultrafilter on $\mathcal{P}(\mathbb{N})$ [31]. The existence of the ultrafilter is a non-constructive weakening of Choice [34].

in various set theories. For the purposes of exposition, I sketch only an informal construction.

Consider the following process. Start with T, the complete theory of $\mathbb{R}$. Then, stipulate that according to the new theory there is a fixed positive infinitesimal, $*$. Call this set $\mathbb{R}_1 := \mathbb{R} \cup \{*\}$. Then close under (e.g.) addition and multiplication by $\mathbb{R}$, so the new set contains elements like $5 + *$ and $\pi \times *$. These elements behave 'as expected', so $5 < 5 + * < 5 + \frac{1}{n}$ for all $n \in \mathbb{N}$. Call this $\mathbb{R}_2$. It contains infinite numbers, e.g. $\frac{1}{*}$. But, $\mathbb{R}_2$ isn't closed under addition or multiplication (by elements of $\mathbb{R}_1$), it contains $*$ but not $* \times *$. So close again under addition and multiplication, etc., by $\mathbb{R}_2$. Call this $\mathbb{R}_3$. But $\mathbb{R}_3$ isn't closed. So take the infinite closure $\overline{\mathbb{R}} = \bigcup_{n=1}^{\infty} \mathbb{R}_n$. $\overline{\mathbb{R}}$ now is closed under addition and multiplication (by a compactness argument).

The standard numbers form a copy of $\mathbb{R}$ in $\overline{\mathbb{R}}$, but there is much besides, e.g. $5 + (*)^{14}$, or $\frac{4+*}{e^{\pi}+(17.1\times *)}$, finite such numbers are close to members of the copy of $\mathbb{R}$ in $\overline{\mathbb{R}}$.

Definition 2. *The standard part, $st(r)$, of a finite $r \in \overline{\mathbb{R}}$ is the unique closest real to it. Non-standard numbers are 'approximately equal' if their difference is 0 or infinitesimal. If x is approximately equal to y, write $x \approx y$. So $st(r) \approx r$.*

We can now check that $\overline{\mathbb{R}}$ 'works like' $\mathbb{R}$. The basic properties of the structure are preserved, e.g. the expected commutativity, associativity, and distributivity of addition and multiplication. These hold by construction. But analysis concerns many other functions. To enable discussion of these, each function f on $\mathbb{R}$ must have a 'natural extensions' f^* on $\overline{\mathbb{R}}$. The existence of extensions can be axiomatic (in non-constructive accounts, [40]:9) or can be proved in a specific construction, for example via ultrapowers ([31]:10). Some extensions are obvious, e.g. $(x^2)^* = ((x)^*)^2$, whilst $sin(x)^*$ may be less obvious. But these extensions are not *mysterious*, just as if one had only seen $sin(x)|_{\mathbb{Q}}$, the irrational values of $sin(x) : \mathbb{R} \to \mathbb{R}$ aren't obvious, but this does not make the $\mathbb{R}$-function unexplainable.

Complicated properties of function can be elegantly proved in NSA:

Definition 3 (Continuity [26]:43)**.** *A function f is continuous at $r \in \mathbb{R}$ iff for every $h \in \overline{\mathbb{R}}$ such that $h \approx r$, $f^*(h) \approx f^*(r)$.*

This definition 'makes sense', i.e. it agrees with the standard definition of continuity. For example:

Example 4. $f(x) = x^2 + x + 1$ *is continuous at* $x = 1$

Proof. $f^*(1) = f(1) = 3$. The required property holds if $h = 1$. Suppose $h \neq 1$. Then $h = 1 + i$ for an infinitesimal i. So $f^*(h) = (1+i)^2 + (1+i) + 1 = 3 + i^2 + 3i$. So $f^*(h) - f^*(1) = i^2 + 3i = i(3+i) \approx 3i$, which is clearly infinitesimal. So $f^*(h) \approx f^*(1)$. □

This generalises in the obvious fashion to a simple 'arithmetical' way to do analysis. In fact, $\overline{\mathbb{R}}$ is an elementary extension of $\mathbb{R}$ ([5]:78). So, for any finite collection of (standard) reals C, and any sentence ϕ in $\mathcal{L}(\mathbb{R})$, $\phi(C)$ is true according to $Th(\overline{\mathbb{R}})$ iff $\phi(C)$ is true according to standard analysis: the two theories say exactly the same (first-order) things about the reals. This theorem is known as the transfer principle, and initially motivated NSA. So NSA is distinctly *un*like classic examples of enriched mathematical ontologies, like large cardinal extensions of set theory. These increase the proof-theoretic strength of the set theory. Meanwhile, NSA is (first-order) conservative:

Theorem 5 (Transfer Principle [31]:21)**.** *A simple sentence* ϕ *of* $\mathcal{L}(\mathbb{R})$ *is true in* $\mathbb{R}$ *iff the non-standard translation* ϕ^* *is true in* $\overline{\mathbb{R}}$.

Definition 6 ([31]:13)**.** *A* simple sentence *in* $\mathcal{L}$ *is either atomic or of the form* $\forall \hat{x} \Big(\bigwedge_{i=1}^{k} \overline{P_i}(\overline{\tau}_i) \to \bigwedge_{j=1}^{h} \overline{Q_j}(\overline{\sigma}_j) \Big)$ *where e.g.* $\overline{P}$ *names a relation* P *and* τ *is a term.*

Simple sentences are roughly the bounded sentences of first-order logic. $\overline{\mathbb{R}}$ is not conservative at higher-orders, as second-order sentences have different truth-values.

Example 7. *Every subset of* $\mathbb{R}$ *which is bounded above has a least upper bound.*

This is an elementary fact about $\mathbb{R}$. But it is not true in $\overline{\mathbb{R}}$: consider the set $U = \{h \in \overline{\mathbb{R}} : 0 < h < 1 \wedge h \not\approx 1\}$, for any proposed non-standard lower bound $1 - i$, $1 - i^2$ is a smaller lower bound for U. But this is second-order, so the transfer principle does not apply.

Meanwhile, functions which are *not* natural extensions behave differently than expected. Consider the following 'unnatural' function:

Example 8.

$$D(x) = \begin{cases} 1 & \text{if } x \approx q \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Its restriction to $\mathbb{R}$, $D(x)|_{\mathbb{R}}$, is Dirichlet's Function, which is nowhere-continuous on $\mathbb{R}$. But for all $h \approx x$, $D(h) \approx D(x)$. This makes it seem like $D(x)$ *is* continuous. The explanation here is that $D(x)$ *is not* the natural extension of the Dirichlet Function, it is another beast entirely. $(D(x)|_{\mathbb{R}})^* = 1$ for $x \in \mathbb{Q}$, hence at every hyperrational; but a hyperrational approximation of i is approximately equal to i, so $(D(x)|_{\mathbb{R}})^*$ takes different values infinitely close to i, so is not continuous. Just like any other extension of a theory, the domain of $\overline{\mathbb{R}}$ gives scope for all kinds of weird functions. These are certainly in the space $\overline{\mathbb{R}}^{\overline{\mathbb{R}}}$. But they *are not* functions in $\mathcal{L}(\overline{\mathbb{R}})$. In ordinary contexts, NSA is limited to talking about the *natural extensions* of $\mathbb{R}$-functions, so such examples do not arise. So, NSA *does not* use full second-order logic. But this kind of restriction on the kinds of functions of $\overline{\mathbb{R}}$ can be motivated, see §3.2.

Henceforth, **R** refers to some model of NSA. The above construction may be slightly less philosophically motivated than the ultrapower construction. For example, CH guarantees that the ultrapower construction of **R** is unique up to isomorphism (see §6.2). It also seems more *explanatory* than compactness arguments: it makes one non-constructive assumption (the existence of a free ultrafilter on $\mathcal{P}(\mathbb{N})$), which gives more of a grasp on what the model is like ([31]:25), rather than a 'brute fact' that there is such a model.[3] However, the majority of the discussion applies to each construction equally, so I use **R** ambiguously, highlighting arguments which rely on one particular construction.

3 Applications

> There are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.
>
> Gödel [19]:iii

Here I outline several important metrics where **R** seems to win hands-down. I first sketch pedagogical advantages. Some, like Halmos, claim

[3]Intuitionistically permissible models have been constructed [49].

that the transfer principle suggests that NSA is merely a (very useful) methodological trick ([20]:204, [4]). However, in §3.2, I argue that $\mathbb{R}$-continuity is a useful property, but that this does not show that $\mathbb{R}$ best captures the intuitive notion continuum. In §3.3, I describe some advantages in explanatory power, and explore the great *mathematical* power of NSA. These seem to suggest that NSA is a more appropriate representation of mathematical structures, principally *continuous* structures. This is sometimes known as Henson and Keisler's indispensability argument ([4]:4). The pedagogical advantages can be read in this light: it is *unsurprising* that NSA is useful for students, given that it better captures the intuitive concepts.

3.1 Pedagogy

Pedagogically, NSA seems to align more correctly with student intuitions about analysis, allowing familiar arithmetical processes into analysis. The available data suggest that students are better able to manipulate NSA (e.g. for finding limits), and, more subjectively, that it improves *understanding* of analysis ([9]:190).

A 1976 study tested 136 students (half taught standardly, half taught NSA) of similar initial ability on basic analysis [57].[4] The differences were significant. One question is discussed in detail, where students were asked to find the limits at the discontinuity of a split definition function (something more analytic than most basic calculus). The results corroborate the pedagogical claims very strongly. Less than 7% of attempted proofs by the control (standard) group were satisfactory, whilst more than 92% of the non-standard arguments were satisfactory, suggesting good ability to *calculate*. Meanwhile, about 32% of control students did not attempt the question, compared to only 6% of the NSA group; this suggests improved understanding. Clearly, test results are affected by many factors, but the opinions of the teachers were very similar:

> the students learned the basic concepts of the calculus more easily, proofs were easier to explain and closer to intuition, and most felt that the students end up with a better understanding of the basic concepts of the calculus. (Dauben [9]:191)

[4]This study records teachings from a book which uses the *axiomatic* approach to **R** ([40]:27). A 2017 paper recounts very similar conclusions about the understanding of more recent students ([37]:8).

This was supported by overall results of the instructor questionnaires ([57]:374), with near unanimous agreement that NSA fitted better with the students' intuitions, and that the students found the concept of 'infinitely small' natural. This suggests that NSA was not merely a useful methodology, but really captures the intuitive mathematical notion of 'continuity'.

There are limitations to this data. This study was based on a relatively small group of American analysis students. Some corroborating undergraduate data are in [38]. Still, one should be careful not to over-generalise. The study included questions about $\mathbb{R}$ rather than $\mathbf{R}$, leaving NSA open to claims that it is only instrumentally useful.

However, this focus on $\mathbb{R}$ seems to be an artefact of designing a test which the control group could answer. One might hope for further data investigating e.g. whether teaching of standard analysis relies on non-standard intuitions already.[5]

A more worrisome issue is that these students might have only understood the *standard structure* embedded in $\mathbf{R}$. It's unclear whether the student grasped the subtlety of moving between functions and their natural extension, if, for example, they could naturally extend an arbitrary function f to f^*.

There is insufficient information to establish whether they did fully grasp this subtlety. But it seems that familiarity with both versions of analysis prepares students better for analysing continuous phenomena. This suggests that care with natural extension, like care with a function's domain, is something that must be reinforced to students as part of a pedagogical approach which otherwise benefits them.[6] Moreover, the absence of data is not data of absence: given that the available data suggests that students are better able to calculate and understand NSA, and that instructors heavily stressed the match-up between NSA and intuitions, it seems as though the scales tip towards NSA pedagogically.

A linked point is that non-standard methods are used in mathematical applications, particularly in physics and economics ([66]:12). Physics appeals to infinitesimals *anyway* ([16]:292), perhaps because the functions

[5] Anecdotal data tempts me to believe this occurs.

[6] Checking properties of natural extensions (e.g. the discontinuity of $D(x)$) may be easier than explicitly characterising them. But analogously, checking properties of certain $\mathbb{R}$-functions is easier than defining them primitively (e.g. $sin(x)$).

encountered in physics are well-behaved. Applications include the mathematical foundations of Feynman path integrals, quantum field theory, and solutions to Sturm-Liouville problems ([66]:12). These are notably all formalisations of continuous processes. Rather than being radically revisionist, claiming that physics is *really* using $\mathbb{R}$, naturalism would take physics at its word. This suggests that there is substantial instrumentalist motivation for $\mathbf{R}$ as a foundation for continuous applied mathematics.

One complication is that physicists may profess to using infinitesimals as a short-hand, for ε-δ management. However, this may be based on mathematicians' insistence that infinitesimal methods are 'bad practice' (thanks to standardist hegemony). It seems likely that against a background of a well-founded infinitesimal analysis like NSA, physics would continue to use infinitesimals, perhaps no longer professing to use them as a shorthand (after all, physics doesn't strictly require $\mathbb{R}$, but continues to use it).

3.2 Primacy of $\mathbb{R}$

One might criticise these arguments the pedagogical advantages of $\mathbf{R}$ on the grounds that they seemingly give $\mathbb{R}$ primacy. The tests notably ask questions about $\mathbb{R}$, not $\mathbf{R}$. This seems to place $\mathbb{R}$ 'at the centre', as a privileged bunch of entities within $\mathbf{R}$.

The first comment to make is that $\mathbf{R}$ has a richer higher-order theory ([4]:4), so the primacy of $\mathbb{R}$ is only first-order. More substantially, we might desire to *maintain* the (first-order) results of real analysis. When a number system extends, e.g. from $\mathbb{Q}$ to $\mathbb{R}$, some class of '$\mathbb{Q}$-facts' should stay the same. For $\mathbf{R}$, there is seemingly independent reason to extend $\mathbb{R}$ (e.g. to better capture continuity). Meanwhile, the facts about the subsystem $\mathbb{R}$ should stay the same.

However, this suggests that we *really* want results about $\mathbb{R}$, not $\mathbf{R}$. This seemingly motivates adopting $\mathbf{R}$ as a piece of ideal mathematical methodology, rather than committing to a new conceptualisation of some intuitive structure (the continuum). The same idea comes up in several forms, e.g. with functions. The (legitimate) $\mathbf{R}$-functions are the natural extensions of $\mathbb{R}$-functions. Generally, we ask about $\mathbb{R}$-continuity, *not* $\mathbf{R}$-continuity.

The detractor would explain this by saying that $\mathbf{R}$ is (only) *instrumentally useful*. This option is available to pure instrumentalists or fervent formalists, like Nelson, but is not obviously sufficient for the ordinary realist. The ordinary realist needs something stronger, that **R** *really* is a good candidate for the continuum. One must explain *why* analysis principally concerns the $\mathbf{R}^{\mathbf{R}}$-subspace $\mathbb{R}^{\mathbb{R}}$.

But the claim that mathematics concerns *only* $\mathbb{R}^{\mathbb{R}}$ is too strong: there are substantial mathematical uses for **R** ([27]:378), discussed in the next section. Pedagogically, any primacy of $\mathbb{R}$ seems an artefact of designing a test suitable for the control group. A hyperreal-focused syllabus may have been simply too radical given probable restrictions, like outcome requirements for the students. The pedagogical comparison would only be fair if the course designers and instructors had genuinely free choice on the content of their analysis courses, even at the basic level.

It seems like the real question is not why is $\mathbb{R}$ primary, but why is $\mathbb{R}$-continuity interesting. This can be answered by analogy with other 'nice' substructures that mathematics tends to be interested in. Even standard analysis is principally interested in continuous (or integrable or n-differentiable) functions, *not* the whole space $\mathbb{R}^{\mathbb{R}}$. **R** may be the right notion of a continuum, whilst $\mathbb{R}$-continuity is particularly interesting. This can be unpacked with a *split space* of relations on **R**, where one class of relations are the ordinary natural extensions which e.g. obey the transfer principles; meanwhile, the other (arbitrary) relations are further removed from $\mathbb{R}$ ([29]:663). This allows talk about **R**-continuity, etc.

This might also be explained in terms of *expansion*. The thought goes that standard analysis is true about its domain, but there are reasons to think it is not all-encompassing as an account of continuity. Nevertheless, extensions should agree with $\mathbb{R}$ so far as it goes. This explains the interest in *natural extensions* of $\mathbb{R}$ functions. Trigonometric functions can be understood over $\mathbb{Q}$, that does not mean that considering their natural extensions to $\mathbb{R}$ prioritises rational analysis over real analysis, it just shows that the initial functions were somehow *right*. We can still look at non-natural extensions of $\mathbb{R}$ functions, but they may well have weird properties - just like discontinuous $\mathbb{R}$-functions can have weird properties.

Alternatively, one might say that $\mathbb{R}$ isn't primary as much as $\mathbb{R}$-continuity is well-behaved. The importance of $\mathbb{R}$-continuity could then be explained naturalistically. Applications in natural science are overwhelmingly $\mathbb{R}$-continuous. Meanwhile, I present arguments to think that **R** better captures the intuitive notion of a continuum. So, naturalism justifies the importance of $\mathbb{R}$ as a well-behaved subsystem, whilst more fundamental mathematical and philosophical arguments are required to unpack continuity in the abstract. The mathematical uses of **R** fit nicely with this picture, as they tend to be more complex (advanced parts of e.g. probability theory).

It's just not clear that $\mathbb{R}$ is *problematically* primary. Instead, $\mathbb{R}$ can be thought of as a nicely behaved subsystem of **R**, which is useful for, e.g. scientific applications. However, the mathematical uses which are *only* interpretable in **R** (and those of its rich higher-order theory) show that $\mathbb{R}$ isn't *always* primary. These uses are the topic of the next section.

3.3 Mathematics

NSA has considerable mathematical motivation, both methodologically and explanatorily. I describe NSA explanations and uses in convergence comparisons, asymptotics, combinatorics and probability theory, and outline some of Tao's non-standard mathematics. These also suggest that NSA may be better at capturing the intuitive continuum.

A strength connected to the pedagogical advantages of NSA is its ability to better *explain* mathematical phenomena. Whilst $\mathbb{R}$ fares well in recording measurements, its 'static view' is ineffective at representing the *rates* of convergence, being blind to "how we arrive at the recorded measurement" (Fenstad [16]:289). The sequence $(2^{1/n})$ clearly converges less quickly than $(2^{1/n^n})$, though $\lim_\infty 2^{1/n} = \lim_\infty 2^{1/n^n}$. These rates of convergence *can* be formalised in standard analysis, by comparing various m-derivatives of the functions $2^{1/x}, 2^{1/x^x} : \mathbb{R} \mapsto \mathbb{R}$. But this seems very complex, it's not obvious why we should *need* to consider derivatives of such $\mathbb{R}$-functions. Meanwhile, rates of convergence are captured very naturally by NSA: for any $n \in \mathbb{N}^*$, $2^{1/n^n} \leq 2^{1/n}$, but the infinite indices are informative about convergence properties, whilst the standard indices are not: the terms at just one infinitely index are enough to distinguish their rates of convergence.

NSA is also better at explaining complex asymptotics ([16]:289):

> It is common practice among asymptoticists to speak informally about "fixed" numbers, to be distinguished from numbers depending on a "large" or "small" parameter. Nonstandard analysis legalizes this matter of speaking. (Van den Berg [65]:iv.)

'fixed' becomes 'standard', etc. NSA provides a natural way to interpret 'order', or more complex phenomena like a sequence (s_n) being an asymptotic expansion for a function f (where $\lim_{\infty}(f - \sum_0^n s_i\phi_i)/\phi_n = 0$, [45]:139). In Robinson and Lightstone's words, non-standard criteria better represent asymptotic expansions as they are "suitable for *all* asymptotic sequences $[(x^{-s_n})$ where (s_n) strictly increases] simultaneously" ([45]:156). Naturally capturing asymptotic expansions is an example of NSA's power to describe approximations (see §6.3).

This power of NSA in explaining certain phenomena should be unsurprising: $\mathbf{R}$ is a richer pointset of the line ([16]:298). But these are genuine mathematical phenomena which we should hope to explain, which motivates the acceptance of NSA by inference to the best explanation. This suggests that $\mathbf{R}$ is *not* merely a useful method, but a *better* way to capture certain mathematical phenomena. Moreover, asymptotic behaviour seems to be a kind of *continuous* phenomena. So $\mathbf{R}$ explains some continuous phenomena better than $\mathbb{R}$. This indicates that $\mathbb{R}$ might not be the only (or best) way to capture the intuitive notion of continuity.

NSA has further uses in combinatorics. In terms of theoretical economy, NSA can unify the disparate uses of ultrafilters in combinatorics [11], as the ultrapower construction of $\mathbf{R}$ fixes a unique free ultrafilter on $\mathbb{N}$. Other applications aren't linked to a particular construction, and seem genuinely more explanatory, e.g. the following is standardly proved as a corollary of Ramsey's colouring theorem:

Theorem 9 ([19]:222)**.** *If $\langle P, \leq \rangle$ is an infinite poset, then P contains an infinite strictly monotonic sequence, or an antichain.*

The standard proof does not establish the colouring of the infinite subset $Q \subseteq P$ ([19]:223). Meanwhile, the independent non-standard proof gives more of a sense of the construction of a concrete subset, by inductively defining an antichain under the assumption that there is no such sequence. These methods are not standardly available.

This verges on a *mathematical* rather than *explanatory* reason, as we expect the more informative of two mathematical explanations to be 'right'. Even if this is disputed, there are outright mathematical motivations for NSA (see [33], [66]). These do not merely simplify already established proofs. Instead, whole areas rely significantly on non-standard methods. Notable examples are probability theory and stochastic analysis [1]. Even if standard translations to these results can theoretically be found, they might be 'humanly incomprehensible' ([4]:5). In short, though theoretically equal, NSA is *practically* stronger than standard analysis. Here, Henson and Keisler present an indispensability argument:

> There are several results in probability theory whose only known proofs are nonstandard arguments, which depend on saturation principles ([27]:377)

Definition 10 (Saturation [47]:138)**.** *$\mathcal{M}$ is κ-*saturated *iff for all $X \subseteq \mathcal{M}$ where $|X| < \kappa$: if any finite satisfiable set of sentences with parameters from X is satisfiable. $\mathcal{M}$ is* saturated *if it is $|\mathcal{M}|$-saturated.*

$\mathbb{R}$ is not saturated, e.g. the set $\varepsilon(c) := \{0 < c < \frac{1}{n} : n \in \mathbb{N}\}$ is finitely satisfiable but not satisfiable. Meanwhile, $\mathbf{R}$ *is* saturated ([19]:138). Some specific non-standard results (using saturation) include implications of viewing Brownian motion as a "random walk with infinitesimal steps" ([41]:v). The fact that the *only* proofs of these various results are non-standard is good evidence to suggest that NSA is the appropriate way to capture these phenomena. Moreover these phenomena have a continuous nature ([30]:387).

Henson and Keisler's reading of this is that NSA *really* has greater proof strength at higher order, noting the difference between their second-order theories ([27]:377). In general, NSA "uses a larger portion of full ZFC than is used in standard mathematical proofs" (Henson & Keisler [27]:378). So long as one is not squeamish about higher-order arguments, then $\mathbf{R}$ has a richer theory with greater mathematical utility.

Perhaps the most influential modern advocate of the mathematical significance of NSA is Terrence Tao, who routinely uses non-standard methods [59]. I outline some uses, and encourage readers to investigate this rich vista. Sometimes he provides more intuitive proofs to previously established theorems e.g. Szemerédi's theorem (in arithmetic combinatorics), Siegel's theorem on integer points (in Diophantine equations), or in establishing polynomial bounds. Other results seem to be entirely new, e.g. almost quantifier elimination [62], and an implication of Szemerédi's theorem [60].

Other points are more systematic, with significant philosophical interest, notably Tao's position that ultraproducts are a bridge between discrete and continuous analysis [63], and his argument for viewing $\mathbf{R}$ as the elementary closure of $\mathbb{R}$ [61] (see §7). His use of non-standard methods suggest that he should accept $\mathbf{R}$ as a good candidate continuum. Much could be said about Tao's uses of NSA, but the rub is that rather than being 'ε-δ management' [58], they are *fundamentally* non-standard uses, which are not obviously interpretable standardly.

4 Continua

'Continuum' is a mathematically loaded term. 'Linear continua' typically refers to dense, Dedekind complete orderings, whilst topologists call (unordered [69]:57) compact, connected, Hausdorff spaces 'continua' ([68]:203). The aim here is to capture the *intuitive* notion of continuity, especially that related to the geometric line. Though geometry has no inherent coordinate structure, the first goal is to capture and analyse it using numerical structures. Meanwhile, there is an intuitive notion of continuity, which has some life beyond its geometric interpretation. The second goal is to capture this analytically.

A common, if implicit, assumption is that continuity is best captured by Dedekind completeness. $\mathbb{R}$ is Dedekind complete whilst being somehow nice, e.g. being a field. Hence(!) it is *the* model of the continuum. Some go further, asserting that $\mathbb{R}$ captures the geometric line. In §5, I reconstruct these arguments for these beliefs, and analyse them in §6. They are found wanting. There are also reasons to think that Dedekind completeness is *unnecessary* for intuitive continuity. So, the candidate for continuity which endorses $\mathbb{R}$ may not be credible.

In §7, I gesture towards alternative ways of capturing continuity, for which $\mathbf{R}$ is a promising representative, thanks e.g. to strong closure properties. Then, as $\mathbf{R}$ succeeds over these candidates in certain significant metrics, this suggests that we should be open to $\mathbf{R}$ as a candidate for (a or) *the* continuum.

One explanation of the differing benefits of the systems is that $\mathbb{R}$ is *one way* of modelling the continuum, and $\mathbf{R}$ is another. They create different pointsets, and imbue the geometric line with a different coordinate structure which might be used in different settings. This might motivate a policy of pluralism, rather than *replacement* of $\mathbb{R}$ with $\mathbf{R}$.

The pluralist approach to coordinatising the intuitive *geometric* line is already occupied ([17]:60): Some NSA practitioners allow for *multiple* coordinatisation, rather than taking $\mathbf{R}$ to be "a model of *the* continuous straight line of geometry" (Ehrlich, [14]:23). Meanwhile, Fenstad claims that mathematicians are rarely pluralist: "most mathematicians would claim that the 'real' points ... exhaust the geometric line" ([17]:57), i.e. hold the following:

Axiom 11 (Cantor-Dedekind Postulate [12]:49)**.** $\mathbb{R}$ *is order-isomorphic to the geometric line.*

It seems like geometric arguments are philosophically relevant in establishing the best candidate for the arithmetic continuum, but these make an extra step. Yes, the geometric line should be coordinatised by the arithmetic continuum, but these are still *distinct objects*. Unlike the geometric line, the arithmetic continuum has a coordinate structure, direction, and ordering (though the geometric line has some notion of 'betweeness'). Some might go further and say the geometric line is not made up of points, whilst the arithmetic continuum is. Either way, we should be open to the possibility that there is a best candidate arithmetic continuum, but that it need not *always* best coordinatise the geometric line.

5 Dedekind Gaps

> It is a common belief of mathematics that the geometric line is a pointset which can be coordinatized by the set of real numbers
>
> Fenstad, [16]:289.

We have a rough grasp on how the discrete and continuous differ, as shown by our ability to rank processes by discreteness or continuity. Iteration (e.g. adding a unit) seems very discrete, whilst movement along e.g. a sine wave seems continuous. Fenstad draws out the distinction between intuitions that have their root in *counting and labelling* against those with their root in *measurement* ([17]:57). We intuitively grasp this informal *continuous number system*, the continuum. A first approximation of continuity might be the existence of a number between any two numbers:

Definition 12 (Between-continuity)**.** *A number system $\mathcal{N}$ is* between-continuous *if $\forall x < y \in dom(\mathcal{N})$, $\exists h \in dom(\mathcal{N})$ such that $x < h < y$.*

This property has an alias, density. But, before familiarisation with density, it might have been a candidate definition of continuity. So, the first candidate continuum might be $\mathbb{Q}$, which is an ordered field, between-continuous, and 'closed under measuring its ratios'.

But this isn't *full* enough to be a good candidate continuum, because there are certain naturally occurring measurements which correspond to no number in $\mathbb{Q}$. The *arithmetic* continuum is meant to capture the intuitive continuous number structure, one that coordinatises, i.e. measures, the geometric line.

> [When measuring] we soon experience that ... we need an extension of the notion of number to measure e.g. the diagonal of the square. (Fenstad [17]:57.)

The typical next step is to *complete* this ordering, by filling in certain 'gaps'. The question is: which gaps are relevant?

A first stab might be *algebraic* gaps. For example, rational polynomials can be defined which have no roots in $\mathbb{Q}$, e.g. $f(x) = x^2 - 2$. These seem to be gaps that one would naturally expect not to occur in a continuum. So, continua ought to be *real closed*.

Definition 13 (Real Closed Field)**.** *A totally ordered field $\langle \mathbb{F}, < \rangle$ is real closed iff every positive element of $\mathbb{F}$ has a square root in $\mathbb{F}$ and any polynomial of odd degree with coefficients in $\mathbb{F}$ has at least one root in $\mathbb{F}$*

However, such algebraic properties don't recommend a unique candidate continuum. $\mathbb{R}$ and $\mathbf{R}$ are real closed, and there are real closed fields of any cardinality ([67]:426). $\overline{\mathbb{Q}}^{alg}$ might even be more justified than $\mathbb{R}$ or $\mathbf{R}$ as it can be built directly out of $\mathbb{Q}$, for which there is already need. Indeed, if algebraic closure were crucial, it might recommend $\mathbb{C}$ - but this cannot be required to capture the *continuum*: it is a classic case of a *plane*, not a line. It seems that algebra alone cannot capture continuity. If it is simply *necessary*, both $\mathbb{R}$ and $\mathbf{R}$ fit.

But there appear to be 'stronger' notions of gap which should be eliminated. Further extensions are motivated by measurement, e.g. π, the circumference-diameter ratio ([43]:251). Much more generally, the common belief seems to concern model-theoretic gaps in the ordering, as seen from some particular model-theoretic perspective.

Definition 14 (Dedekind-gap)**.** *A dense linear ordering O has a* Dedekind-gap *wherever O is an initial segment $S \neq \emptyset$ of O has no least upper bound.*

To justify the common belief, one must then argue that the gap itself should be added as a new number, and so 'fill-in' $\mathbb{Q}$ (or $\overline{\mathbb{Q}}^{alg}$) to $\mathbb{R}$:

Definition 15 (Dedekind completeness)**.** *An ordered field $\mathbb{F}$ is Dedekind complete iff it has no Dedekind-gaps.*

In this sense, $\mathbb{R}$ completes $\mathbb{Q}$. Finally, the proponent of $\mathbb{R}$ would assert that Dedekind completeness *just is* continuity. Dedekind-gaps are the *only* important notion of gaps when it comes to continuity. So when these gaps are filled, the resulting number system is continuous. The claim is that *this is all continuity means*.

This makes $\mathbf{R}$ discontinuous, as it is not Dedekind complete. For example, the bounded set $[0, 1)\backslash\{h \in \mathbf{R} : h \approx 1\}$ has no least upper bound. This problem remains if Dedekind completeness is merely *necessary* for continuity.

Taking Dedekind completeness to be necessary for continuity also means *denying* that there are incomparables in a continuum, i.e. affirming that continua are Archimedean:

Axiom 16 (Archimedean Property)**.** *Let $\mathcal{N}$ be a number system for which $\mathbb{N}$-multiplication is defined. For all $x, y \in \mathcal{N}$, $0 \leq x \leq y$ there is an $n \in \mathbb{N}$ such that $nx \geq y$.*

Dedekind completeness can be analysed in these terms:

Proposition 17 ([7]:17)**.** *An ordered field $\mathbb{F}$ is Dedekind complete iff it is Archimedean and sequentially complete.*

So defining continuity as Dedekind completeness supposedly delivers a further blow to NSA:

Proposition 18. $\mathbf{R}$ *is not Archimedean.*

Proof. Let $*$ be a positive infinitesimal. $0 < * < 1$. But, for all $n \in \mathbb{N}$, $* \cdot n < 1$. □

Some claim that continua are *ipso facto* Archimedean. Commentators have read Cantor in this light, he certainly opposed "the Cholera-Bacillus of infinitesimals" ([67]:425). He seemingly viewed the Archimedean property as a self-evident truth rather than an arbitrary axiom: "he refused to regard it as an axiom at all. Instead, he argued it followed directly from the concept of linear number" (Dauben [10]:235).[7]

Finally, some might claim the book closes on NSA with Dedekind's (second-order) categoricity result. Early philosophy of mathematics hoped for the categoricity of various theories, i.e. having exactly one model up to isomorphism. Model-theoretic advances show that categoricity results are typically either unavailable or not first-order ([5]:§*B*). However, Dedekind completeness is already a second-order notion. According to *full* second-order semantics, $\mathbb{R}$ is categorical:

Proposition 19 (Dedekind ([5]:155))**.** *The second-order axiomatisation of arithmetic is categorical.*

Corollary 20. *The second-order axiomatisation of* $\mathbb{R}$ *is categorical.*

This would rule out the possibility of NSA as a model of ordinary analysis. Some might claim that is NSA's death-knell. Next, I argue that even though we seem to need *some* higher-order notions, there are more fundamental reasons to think that (the theory of) $\mathbb{R}$ does not capture the intuitive continuum: the problem is the *characterisation*, not the logic.

6 Not Dedekind Gaps

Here I mount a response, arguing that continuity may not be best captured by Dedekind completeness. Again, the target is the *intuitive* notion of continuity. I first question the second-order nature of Dedekind completeness. However, the model-theoretic perspective required for NSA seems to undermine first-orderism, or at least requires a wide model-theoretic perspective. Instead, I question whether Dedekind completeness is even a desideratum, with several concerns about how well $\mathbb{R}$ respects the intuitive notion of continuity, e.g. understanding how points can be 'close but distinct'. I then sketch certain objectionable implications of Dedekind completeness, principally that it begs the question. Instead, I propose endorsing a different notion of gap as crucial for (dis-)continuity.

One argument which can be quickly dismissed is that measurement uniquely recommends $\mathbb{R}$. We only seem disposed to intuitively measure

[7] See [10]:350.67 for historical sources.

rational numbers, and some amount of algebraics, but little else.[8] *Empirical* measurements use $\mathbb{Q}$ at most:

> from the point of view of the empirical scientist ... all measurements are recorded in terms of integers and rational numbers, and if our theoretical framework goes beyond these, then there is no compelling reason why we should stay within an Archimedean number system. (Robinson [54]:282).

Much thinner models seem sufficient for such purposes. Both $\mathbb{R}$ and $\mathbf{R}$ go beyond this call of duty.

In fact, $\mathbb{R}$ only helps measures structures which *already* 'build-in' $\mathbb{R}$, e.g. real analysis. Equally, we might measure mathematical quantities against $\mathbf{R}$ supposing it is 'built-in' instead. This is an early indicator that justifications for $\mathbb{R}$ may be circular.

More generally, we start with parallel stories about the motivation and construction of the two fields. $\mathbb{Q}$ and $\overline{\mathbb{Q}}^{alg}$ are somehow messy. An idealisation tidies this mess. $\mathbb{R}$ is an idealisation of the structural data given by our intuitions, but $\mathbf{R}$ is too.

6.1 Order!

One might oppose Dedekind completeness and categoricity because they are second-order, arguing that there are reasons to reject second-order notions outright. However, I outline why NSA requires a wide model-theoretic perspective, making universal first-orderism unsatisfactory for capturing the continuum either way.

One might complain that second-order notions, like Dedekind completeness, are not so much about *numbers* as they are their *subsets*. We don't seem to need a complete understanding of sets of numbers to have a complete understanding of the numbers themselves (e.g. CH is not considered an arithmetical problem [15]). If one had independent reasons to reject second-order axioms, this puts a stop to the characterisation of continuity as Dedekind completeness. Dedekind completeness would not be an axiom, but a 'meta-axiom', and so "for our contemporary logical consciousness and logical conscience a monster" (Hintikka, [28]:331).

[8] Others make even more restrictive claims about intuitive measurement [17].

Moreover, the categoricity claim requires *full* second-order semantics, which seems additionally suspect ([5]:§7&8). Full second-order logic has few of the metatheoretic properties we expect of a logic, notably compactness, and it has substantial set-theoretic baggage (arguably it *is* a set theory [64]), hence some argue that it isn't really a *logic*. Instead, a logic ought to be topic neutral ([52]:398), allowing the charitable analysis of dissimilar set theories.

These problems generalise: *any* logic which delivers this categoricity also suffers:

Proposition 21 ([5]:162)**.** *Any logic which allows for a categorical theory of arithmetic lacks a sound and complete finitary proof system.*

Even the 'nice' ways to capture the intended model of $\mathbb{N}$ (hence $\mathbb{R}$) are metalogical, e.g. as the well-founded, prime, or recursive model ([53]:6). So, one might accept Dedekind's categoricity theorem as a piece of mathematics, but lump it with Brouwer's continuity theorem: true, but silent on the number systems actually encountered. The problem is faulty theoretical assumptions in the logics that prove it.

However, these arguments seem unconvincing on two grounds. Firstly, $\mathbb{R}$ can also be proved categorical with respect to a background set theory, rather than requiring second-order logic. White's response is this causes ambiguity in $\mathbb{R}$. Different set theories can yield non-isomorphic $\mathbb{R}$s, so $\mathbb{R}$ itself is ambiguous ([67]:434). In fact "the notions *set* and the relation $\in$ are characterized by the [ZF] axioms rather weakly" (Pogonowski [53]:4). Using set theory means accepting the possibility of non-standard models of the set theory ([5]:171). But this response isn't decisive, as it also makes **R** similarly ambiguous.

Secondly, supporters of **R** *also* have substantial need for higher-order notions. NSA is richer than standard analysis from a higher-order perspective ([27]:377), which is mathematically advantageous. More substantially, *generating* the non-standard models uses (non-first-order) set-theoretic notions. This is not to say that second-order logic is logic, but that the model-theoretic understanding used in NSA requires something higher-order ([5]:§9). This generates a tense triad of desiderata:

1. Reject full second-order semantics.

   This subverts the categoricity of standard analysis.

2. Use the full power of model (or set) theory.

   This proves the existence of non-standard models of analysis.

3. Reject model theory's judgement that the resulting models are non-standard.

   Accepting this judgement would show that NSA violates full Dedekind completeness.

(1.) and (2.) seem obligatory for the supporter of NSA. For a chance of there being a non-standard model, they accept (1.), so reject full second-order logic. Then, they use various model-theoretic (e.g. compactness) to prove the existence of non-standard models of the theory of $\mathbb{R}$. In a sense, NSA attempts to make good of a (particular) non-standard model.

These concessions seem to force the supporter of NSA to accept a wide model-theoretic perspective ([5]:§9). From this perspective, NSA can be *identified* as non-standard. This also reveals ambiguity in the definitions or constructions of NSA (see §6.2). But most significantly, it means that NSA is fully understood to have Dedekind-gaps.

The solution here is that the implications of the model-theoretic perspective are not as damaging as it seemed. As later discussion suggests, Dedekind completeness may be inappropriate for capturing continuity. Instead, non-standard models seem to do better. So, the defender of NSA can deny full second-order semantics (i.e. accept (1.)), use the full power of model theory to generate **R** (i.e. accept (2.)), *and* defensibly *reject* (3.). They need not insist on first-orderism: we *can* appreciate the differences between NSA and standard analysis without issue, as NSA is independently motivated, in spite of Dedekind incompleteness.

6.2 Ambiguity in R

Before turning to the reasons to reject the necessity of Dedekind completeness for continuity, I outline one potential criticism of the acceptance of the wide model-theoretic perspective, ambiguity, and negate it.

Suppose the supporter of NSA accepts the wide model-theoretic perspective, in order to understand the idea that **R** is non-standard. This perspective makes **R** is ambiguous in three linked ways: categorically, canonically, and iteratively.

Categoricity: Ultrapowers rely on free ultrafilters: these may be different, which would give different, non-isomorphic models for the hyperreals. Similar considerations apply to other versions of NSA (e.g. Keisler's axiomatisation [40]:56). So **R** is not categorical.

Canonicity: Another problem is in terms of standard *constructions*. The competing constructions of $\mathbb{R}$ (Cauchy sequences, Dedekind cuts) are isomorphic, so it doesn't matter which is picked as 'standard'. But the same is not true of **R** [32]. Some might claim there is no good reason to fix one of these dissimilar constructions as standard.

Iteration: The 'hyper' construction can be iterated on **R**, generating a hyper^{n}real hierarchy ([26]:121).[9] Transfer principles apply here too, i.e. they have the same first-order theory. This suggests a scale of very non-standard to very standard models. Preferring **R** over some set of hyper^{n+1}reals is arbitrary, as they have similar philosophical justifications. The hyper^{n}reals also look gappy compared to the hyper^{n+1}reals, as the hyper^{n+1}reals contain a 'hyper^{n}infinitesimal', i, such that $0 < i < \frac{1}{N}$ for each hyper^{n}natural N. Hence, one might argue that *none* is a suitable candidate for the continuum.

Various responses disarm these ambiguity issues, at the cost of denying that there is a *unique* best candidate continuum. A brief reply to the iteration problem comes from simplicity and naturality. The hyper^{n+1}reals have the same mathematical utility as **R** (e.g. they definite $\mathbb{R}$-continuity analogously [26]:122), but **R** is in some sense *simpler* than the hyper^{n+1}reals. It has no iterated monads of infinitesimals, etc. Nor do the hyper^{n+1}reals have substantial advantages over **R** in terms of capturing continuity. So the situation is different from the argument for **R** over $\mathbb{R}$: $\mathbb{R}$ may be simpler, but does not do the work asked of it. Meanwhile, the hyper^{n+1}reals have no substantial mathematical advantage beyond those of **R**, and have no independent support.

This seems to prioritise $\mathbb{R}$ (perhaps unproblematically, see §3.2). However, the real reason that iteration is unproblematic is that it doesn't matter too much which level is chosen. The hyper^{n}reals all 'look' very similar, with similar philosophical, mathematical, and pedagogical properties, advantages, and disadvantages. For example, they are similar from the perspective of applications to continuous phenomena.

[9] Formally: axiomatise **R**, by compactness there is a model with hyper-infinitesimals, etc.

Still, the supporter of some higher, more comprehensive, hyper^{n}reals (or the 'hyper$^\kappa$reals'!) might argue that this apparent similarity is just a result of fixating on the cut between $\mathbf{R}$ and $\mathbb{R}$, and that we might want a fine-grained understanding of a certain set of hyper^{m}reals.

But this doesn't seem critical. One need not be uniquely wedded to a level of the hyper^{n}real hierarchy. Perhaps, ascending this hierarchy improves the approximation of the 'true' geometric line, thought of an object *not* constructed from points. The main point here is that there are reasons to think that *any* level ($n > 0$) is a better candidate continuum than $\mathbb{R}$. This is the first indication of the most sustainable solution: certain metrics show that the best candidates for the intuitive continuum are non-standard, but there may not be *unique* best candidate.

Next is categoricity. One might try to combat the categoricity issue is head-on, by giving unique characterisations of $\mathbf{R}$. I recount two, which rely on dissimilar set-theoretic assumptions, CH and the Inaccessibility Axiom. This victory is limited, as both of these apply only to the ultrapower construction. NSA has other models, so this leaves open the canonicity issue.

Theorem 22 (Croquet-Rudin [23]). *CH implies that there is exactly one free ultrafilter on* $\mathbb{N}$*.*

Corollary 23 ([19]:33). *CH implies that* $\mathbf{R}$ *is unique up to isomorphism.*

The result is unsurprising: limiting the kinds of ultrafilters on $\mathbb{R}^{\mathbb{N}}$ effectively determines what kinds of subsets of $\mathbb{R}$ there are.

If one doubts CH, uniqueness can also be ensured by axioms concerning *superstructures*, of which $\mathbf{R}$ is an example:

Definition 24. $V(X) := \bigcup_{n=0}^{\infty} X_n$ *is a* superstructure*, where* $X_0 = X$ *and* $X_{n+1} = \mathcal{P}(X_n) \cup X_n$

A triple describes the information determining if this is a non-standard model of $\mathbb{R}$:

Definition 25 ([14]:21). *Let* $\mathcal{M}$ *be a model of* $\mathbb{R}$*, and* $* : \mathcal{M} \to \mathbb{R}$ *be a monomorphism.* $\langle V(\mathbb{R}), V(\mathcal{M}), * \rangle$ *is a* non-standard model of analysis *if it obeys the appropriate transfer principle.*

Axiom 26 (Axiom of Inaccessibility). *There is an inaccessible uncountable cardinal.*

A categoricity result can then be proved using this 'height' axiom:

Proposition 27 ([40]:61)**.** *Assume the Axiom of Inaccessibility, and that there is a non-standard model* $\langle V(\mathbb{R}), V(\mathcal{M}), *\rangle$ *which is* $|\mathcal{M}|$*-saturated over* $V(\mathbb{R})$ *where* $|\mathcal{M}|$ *is the first inaccessible cardinal. Then, up to isomorphism, there is a unique non-standard universe.*

So, the proof of the categoricity of the ultrapower model of **R** make different *kinds* of claims about the size of the set-theoretic universe, namely width (CH) or height (Inaccessibility).

Yet the latter uniqueness result seems unconvincing, as the conditions are overly strong, making a substantial demand on the size of the model. This seems unnatural for a guarantee of the uniqueness of the candidate continuum, nor is there a prima facie reason to accept that there *is* such a large model $\mathcal{M}$ (see [13]:35). Worse, these don't protect against ambiguity caused by the non-ultrapower constructions of non-standard models.

The more promising response to the criticism of ambiguity is very different. One can diffuse these ambiguity issues by denying that they were ever really criticisms. Some claim that for *any* concrete area of mathematics, we must put up with non-standard models one way or another, and significantly: this need not be problematic ([48]:127). This resembles the algebraic attitude towards models ([5]:157). One need not think this generally, but in this case, there may be *no best candidate* amongst the models of NSA. Acceptable non-uniqueness may be the solution to the canonicity issue too. Certain models, e.g. the hyper^{n}reals for large n, might better approximate the geometric line. Either way, each model satisfies the alternative conditions for continuity (see §7). This does not undermine the later reasons to think that non-standard models of analysis are systematically better capturing continuity than $\mathbb{R}$.

There may be no (current) indefatigable reason to prefer a particular non-standard model. This is the bullet I would bite. For mathematical purposes, pick a model, but be philosophically open to the possibility that there is no best option. There may be no unique best model satisfying the conditions which better capture intuitive continuity, but that doesn't undermine the arguments for them better capturing the continuum.

6.3 Gaps, Extremality, & Closeness

The advocate of NSA should perhaps accept that there is no unique best non-standard model. But this doesn't prevent them from arguing that (some) non-standard models are better than $\mathbb{R}$ at capturing the intuitive continuum. Earlier requirements, e.g. 'between-continuity' and real closure, don't decisively favour $\mathbb{R}$. This hints at the principle concern about the arguments which favour Dedekind completeness: they may be circular. Arguments which claim to show that Dedekind gaps are the only notion relevant for continuity beg the question. Here, I sketch a few arguments which raise the suspicion in the unscrutinised Dedekind definition of continuity, followed in §6.4 by an extended argument that the Dedekind definition of continuity circularly requires continua to be Archimedean.

A first issue is that Dedekind completeness seems to violate its main motivation: to eliminate gaps. Suppose continuity requires no Dedekind-gaps. Conway's surreals are in a strong sense the *least* gappy possible number system, being the unique homogeneous universal ordered field ([14]:26). So, roughly speaking, they have no gaps from the perspective of *any* suitable number system.[10] But the surreals are not Dedekind complete ([8]:37). They are neither Archimedean (as they contain infinitesimals), nor Cauchy complete (for their appropriate notion of sequence [56]:18). More directly, they have Dedekind gaps where the cuts cannot be defined even in NBG set theory [25].[11] Hence, the surreals would *not* count as gap-less according to the Dedekind definition of gap-lessness. This seems wrong. Dedekind completeness seems not to capture gap-lessness, as forcing these gaps closed eliminates the least gappy number systems. In short: these are the wrong gaps. Hence, Dedekind completeness does not seem essential for candidates for continuity. On these grounds, one might reject the necessity of Dedekind completeness for continuity.

One might think that, as the most comprehensive field, Conway's surreals would be a candidate continuum. But they suffer from independent issues. Firstly, the field is proper class-size, which may be odd for a continuum. To sharpen this, note that the surreals contain all ordinals. So a good understanding of (surreal) analysis would seemingly require a com-

[10] Larger structures violate the field axioms [22].

[11] A bounded surreal subset has a least upper bound *per iteration*, which is subverted in the next iteration [21]. Perhaps reformulating Dedekind completion, e.g. plurally, corrects this. But 'arbitrary cuts' may not be the right way to think of Dedekind completion to begin with [22].

mitment to a truth value for each large cardinal axiom. But such commitments seem totally disconnected from ordinary analysis. There is also a fundamental difference in richness or complexity here: continuity seems to have some fixed level of complexity, whilst the surreals have no limit on their complexity, with each step in their construction being ever richer. Meanwhile, **R** contains only a set-sized number of countable ordinals.

These size concerns become critical given that they make it impossible to carry out ordinary parts of continuous analysis in the surreals. For example, trigonometric functions cannot be extended to Conway's domain [35], [36]: such a function would have a proper class-sized domain, so would ultimately need to be explained in terms of a 'meta-class' which would come 'after the classes'. Hence, for all its advantages, the surreals cannot capture the (intuitive) continuum. Rather, they are something more comprehensive.

The next issue is for Dedekind completeness is that it is unlike other closure principles. Closure axioms are a variety of extremal axiom, which are meant to guarantee the unique characterisation of the models of a theory ([28]:330), either maximally or minimally. Dedekind completeness is *maximal*, asserting "that there is no more comprehensive system of things that satisfies a given series of axioms" (Carnap & Bachmann [6]:68). However, closure axioms are typically *minimal*, e.g. induction ([53]:9),[12] or set-theoretic replacement ([6]:78). This is pretheoretically suspicious: closure axioms are expected to *add* elements, i.e. stipulate a minimum structure, rather than to *bound* the elements, i.e. stipulate a maximum. When taking the closure of a structure under an operation, one typically makes an *additional* (arbitrary) choice to consider the smallest candidate if a concrete example is necessary (e.g. a set's transitive closure), whilst allowing for larger closed structures. Indeed, the choice of the smallest candidate seems motivated by set-theoretic worries (ensuring the candidate isn't class-sized), rather than with ensuring closure. Closure properties themselves don't place an 'upper bound' on the structure, instead one makes an *additional* decision to consider the smallest candidate.

This may not be definitive. Alternatively, one might cash this out like Hintikka ([28]:331): Dedekind completeness (and Hilbertian completeness in geometry) is unrelated to descriptive and deductive completeness, or

[12]This depends on whether induction should be second-order (making it maximal). But excluding the possibility of 'long $\mathbb{N}$' seems to beg the question, as $\mathbb{N}^*$ also satisfies a good formulation of (first-order) induction.

to "the completeness of an axiomatisation of some part of logic" (Hintikka [28]:331). Both of these issues seem to suggest that Dedekind completeness is not a *closure* axiom at all. Instead, it appears to be more like a *set-theoretic* axiom, which perhaps has no direct connection to continuity.

A final issue with Dedekind completeness is its inability to capture *closeness* and *approximatability* ([46]:1),. One might try to phrase continuity in terms of closeness instead of gaps (Dedekind or otherwise). Prima facie, it might seem essential that points in a continuum can be close. However, there is no good fixed threshold of closeness for points in $\mathbb{R}$. Any chosen candidate, say 'a distance of 0.3 apart', is unsatisfactory. The best $\mathbb{R}$ can hope to say is that there is an ($\mathbb{R}$-)arbitrarily close point to any fixed point - i.e., $\forall \varepsilon \in \mathbb{R} \exists q \in \mathbb{R} |p - q| < \varepsilon$. $\mathbb{R}$ cannot talk about *two points* being close, as the points are always some real distance apart. This might be insufficient for continuity. Meanwhile **R** has a satisfactory fixed threshold of closeness: $p \approx h$. This does better in formalising the idea that points in a continuum are 'near to each other', rather than being a real jump away.

Having a good threshold to capture closeness of points allows NSA to better capture approximability of sets ([19]:214). A measurable set S is approximated from above and below by $B_\varepsilon \subseteq S \subseteq A_\varepsilon$ where $\mu(A_\varepsilon \backslash B_\varepsilon) < \varepsilon$. Each standard approximating pair $\langle B_\varepsilon, A_\varepsilon \rangle$ do not approximately it *closely* ($\mu(A_\varepsilon \backslash B_\varepsilon) \not\approx 0$). Meanwhile, non-standard approximations *can* be close to the set ($\mu(A_\varepsilon \backslash B_\varepsilon) \approx 0$). This seems to better capture the notion of (genuine) approximation: NSA can say that a pair $\langle B_\varepsilon, A_\varepsilon \rangle$ approximate S, *simpliciter*, as there is a good threshold for closeness. $\mathbb{R}$ cannot. This also underlies non-standard measure theory, which has substantial utility in describing continuous phenomena ([19]:203, [30]:387, [1]).

The only available response for the standardist seems to be circular: they would need to say that this 'absolute closeness' is the wrong picture, instead, continua only need 'arbitrarily close' points. But it's unclear why continuity should *not* have a threshold for closeness.

Dedekind completeness might be doubted as the definition of continuity. It unsustainably suggests that the surreal numbers are gappy, is somehow unlike other closure properties, and doesn't do justice to another way of thinking about continuity, closeness. Even so, one might argue that Dedekind completeness is *necessary* for continuity. The next section combats this, due to the circularity that arises in *forcing* arithmetic continua to be Dedekind complete.

6.4 The Archimedean Requirement is Circular

To defend Dedekind completeness as the definition (or a necessary condition) of continuity means to hold that any continuous number system *must be* Archimedean. But there are reasons to doubt this. The most substantial issue is that there is no non-circular way to argue for continua being Archimedean.[13] The intent here is to provide an arithmetic continuum, perhaps to coordinatise the geometric line. But there is nothing strikingly Archimedean about continuity or the geometric line. It is totally unrelated to removing gaps, or closeness, etc. Critics have highlighted this as a potential issue in Cantor's thinking:

> had Cantor agreed that the Archimedean property of the real numbers was merely axiomatic, then there was no reason to prevent the development of number systems by merely denying the axiom, ... but to have allowed this would have left Cantor open to the challenge that... his own views on the continuum was lacking. (Dauben [10]:235.)

The question is: why should one believe that *all* numbers in the arithmetic continuum are such that adding them together a number of times will exceed any other number in the continuum? It appears that *mathematical* arguments won't do, as being Archimedean is independent of the other requisite mathematical properties for the continuum (e.g. density, real closure). So the argument must be *philosophical*.

Asserting that continua are Archimedean *because* there can be no 'small' numbers is circular: this just denies that there are incomparables ([67]:243), i.e. asserts the Archimedean property again. Answering that one must "imagine the continuous line, and have numbers correspond to each part of the line" also begs the question, as the proponent of NSA can reply that they *are*, and that those numbers constitute **R**. Cantor's argument that the Archimedean property "follows directly from the concept of linear number" (Dauben [10]:235) also seems non-reductive, as a 'linear number' is just one which corresponds to a point on a continuous line: this does not *justify* the Archimedean property.

The circularity recurs in assertions that the arithmetic continuum must capture the geometric line, with appeals to geometric 'facts', e.g. Hilbert

[13]Being Archimedean is also non-first-orderisable, as it refers to the standard naturals.

Archimedean axiom of geometry. This is non-explanatory, as it relies on the geometric analogue of the Archimedean property, which suffers from analogous circularity.

In this section, I critique an extended defence of the Archimedean property, that $\mathbb{N}$ is a backbone for $\mathbb{R}$. It is found wanting. So there is reason to think Dedekind completeness is not necessary for continuity.

One way to defend the claim that continua must be Archimedean is via the structural advantages this provides. The (standard) integers form a 'backbone' to $\mathbb{R}$. Systematically, define a *yardstick* as:

Definition 28 (Yardstick). *A* yardstick *for a number system $\mathcal{N}$ is a $Y \subseteq dom(N)$ such that (1) for all $n \in dom(N)$ there is a $y \in Y$ which is suitably close to n, and (2) Y is built in a nice way.*

This is clearly vague. Precise witnesses for the conditions can be given for $\mathbb{N}$ as a yardstick for $\mathbb{R}^+$: 'suitably close' means $|y - n| \leq \frac{1}{2}$, and 'built in a nice way' means generation from 0 by iterating the successor function only. However, the naturals *do not* form a yardstick for $\mathbf{R}^+$, as infinite numbers are 'far away' from all (standard) naturals.

One way to make this concern sharp is to say that $\mathbb{N}$ makes $\mathbb{R}$ easy to *visualise*. One might claim that $\mathbb{N}$ grants a clear picture of $\mathbb{Q}^+$ as various ratios. Then $\mathbb{R}^+$ is visualised by filling in the Dedekind gaps. Meanwhile, one might argue that $\mathbf{R}$ is difficult to visualise, e.g. visualising infinitesimals might be difficult.

A first rebuttal is that $\mathbb{R}$ *isn't* so clearly easy to picture either. As Button & Walsh suggest, a continuum of *points* is difficult to visualise *at all* ([5]:89). The pointillist continuum also allows for the construction of difficult to visualise functions (e.g. Peano's space-filling curve [5]:89).[14]

Nor is it clear that visualisation is a very useful judge for mathematics. Being difficult to picture doesn't affect the status of hyperreals as pieces of pure mathematics (try to visualise high dimensional geometry). Visualisation seems not to be a useful criterion.

Instead, $\mathbf{R}$ might be criticised because $\mathbb{N}$ is not its yardstick, whilst an arithmetic continuum 'should' have a backbone. One possible response is that there may be a problem of reference to $\mathbb{N}$. If the *non-standard* naturals

[14]Particular problems with picturing $\mathbb{R}$ are that some ratios are difficult to imagine, and that cuts are difficult to imagine, requiring complex set-theoretic thinking.

are the right account of the naturals, then we may have no resources for pinning down the standard naturals anyway, they are simply a *façon de parler*. For example, *the naturals* mean 'any number that can be reached from 0 by any number of applications of successor', then, with a non-standard metatheory, the naturals pick out $\mathbb{N}^*$. So, to show that they are a yardstick for $\mathbb{R}$, the standardist must assume that the successor construction only reaches as far as $\mathbb{N}$. However, there is a transcendental worry about metaresources here ([5]:206): the same tense triad faced in discussing categoricity in the object language would have its revenge in the metalanguage (see §3.2). No matter what the standpoint, *some distinction* between standard and non-standardness must be accepted. So, it looks like the supporter of NSA should accept the (full) second-order categoricity of $\mathbb{N}$. However, they can still push back against the Archimedean requirement.

A better argument is that $\mathbf{R}^+$ *has* a yardstick, the (hyper)naturals, $\mathbb{N}^*$. Their structure 'looks like' an initial segment of the standard naturals followed by a dense linear ordering of $\mathbb{Z}$s (without end $\mathbb{Z}$s, [39]:13). Certainly $\mathbb{N} \subsetneq \mathbb{N}^*$, but $\mathbb{N}$ and $\mathbb{N}^*$ are otherwise similar, with $\mathbb{N}^*$ doing much of the work asked of $\mathbb{N}$ ([31]:29):

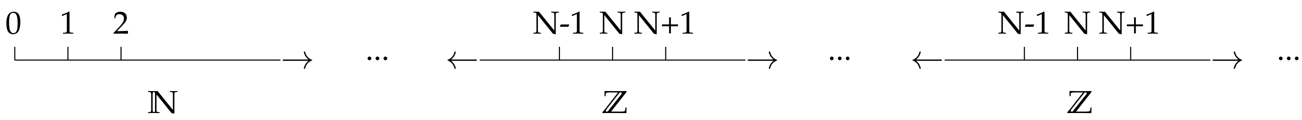

Figure 1: $\mathbb{N}^*$ from the standard perspective

Lemma 29 ([31]:23)**.** *For each $x \in \mathbf{R}^+$, there is a $k \in \mathbb{N}^*$ such that $k \leq x \leq k+1$*

So $\mathbb{N}^*$ satisfies condition (1) of being a yardstick. Moreover, the hypernaturals are still generated by the successor operation, e.g. in the following sense:

Lemma 30 ([31]:23)**.** *For each $n \in \mathbb{N}^*$, $n+1$ is the smallest hypernatural greater than n.*

R also satisifies the non-standard Archimedean property:

Proposition 31 ([19]:35). *For all $x, y \in \mathbf{R}$, $0 < x \leq y$ there is an $n \in \mathbb{N}^*$ such that $nx \geq y$.*

Proof. Let $n \in \mathbb{N}^*$ be such that $n > \frac{y}{x}$, which exists as $\mathbb{N}^*$ is suitably unbounded. Then $nx > y$. □

So, the hypernaturals go a long way to satisfying condition (2) for a yardstick. Hence, **R** has a yardstick, and **R** is filled out from $\mathbb{Q}^*$ as before.

Still, one might point out that the hypernaturals seem not to be generated by *only* the successor function - there are gaps between the $\mathbb{Z}$ segments, which the successor function cannot cross.

More generally, the $*$-map changes the numbers *and* the (ordinary) subsets of numbers. The standard subsets still exist as subsets of **R** *from the outside*,[15] but the defining properties of various subsets might deliver 'larger' subsets. This seemingly means that the (standard) naturals, integers, etc. are *lost*, along with beliefs about them.

The defendant might reply that this should not be surprising. This inability to talk about the standard sets internally is because they are somehow the wrong object, we *should* be considering the new non-standard integers. NSA has *revealed* that the ordinary definitions of analysis had hidden structure. This resembles Hamkins' 'nonstandard axiomatic approach' [24]. We were blind to the full structure of $\mathbb{R}$ (and its naturals). White's explanation is that "because of our relatively impoverished [set theory] we simply did not recognize all of the elements present in, say, the set of $\mathbb{N}$ ... or $\mathbb{R}$ ... - but the unrecognized elements were there all along" ([67]:433). The $*$-map uncovers their richness.

An alternative response is to reassess the niceness of the standard integers. We want to codify the apparent structure of the *naïve naturals* ([18]:39). We can see that the naturals extend far beyond the naïve naturals. We codify these using, e.g., PA, and we desire the most natural or simplest model of PA, etc. Then, says the standardist, the standard integers fulfil this by being 'smallest'.

To counter, although the supporter of NSA can *understand* the standard naturals, there might be cases where smallness is *not* the best measure of naturalness or simplicity. $\mathbb{N}^*$ is in a certain sense natural, in spite of not satisfying full second-order induction. It certainly makes mathematics more efficient, e.g. convergence:

[15] Formally: the pointwise copy of $\mathbb{N}$ is external to **R** ([66]:4).

Definition 32 (Convergence [31]:32)**.** *The sequence $\langle s_n \rangle$ converges to L iff ${s_n}^* \approx L$ for all infinite naturals n.*

Such utilitarian realities do seem to change the subject, they are instead a genuine measure of naturality ([9]:194). So, there's reason to think $\mathbb{N}^*$ is more natural for dealing with continuous phenomena. This seems to support the idea that $\mathbb{N}^*$ can be a suitable yardstick *for a continuum*. As it stands, there seems no clear reason to think that $\mathbb{N}$ is necessary as a backbone for a continuum.

Finally, one can ask why we wanted a backbone at all. Sure, the yardstick is a nice bonus, but the notion of continuity doesn't obviously *require* a backbone. Even if the standard naturals are very rigid, it's not clear necessary that they should measure the intuitive continuum. One thought is that continua are intuitively very homogeneous, where points are essentially interchangeable. So it would be odd to *judge* one candidate over another on the basis that one candidate has a well-behaved subset of distinguished points: distinguished points don't seem to signal continuity.

In summary, the argument against **R** on the grounds of lacking a well-behaved yardstick is circular as: 1.) $\mathbb{N}^*$ is thought badly behaved *because* it is not Archimedean, and 2.) the reasons for wanting a yardstick do not seem connected to candidacy for continuity. This leaves with little justification for the necessity of the Archimedean property. So, Dedekind completeness forcing continua to be Archimedean seems unjustifiable: visualisation arguments fail and claims that $\mathbb{N}$ provides a structure just beg the question. It seems then that Dedekind completeness cannot be necessary for continuity.

7 Elementary Gaps

Based on this evidence, Dedekind completeness may not be necessary for continuity. However, there is something going for it. In this section, I suggest an alternative notion of gaps, *elementary gaps*, which respects some of the motivation for Dedekind completeness, without the above drawbacks.

The Dedekind completeness definition of continuity is primarily justified by the claim that it *closes the relevant gaps*. $\mathbb{Q}$ has Dedekind-gaps, which are filled-in by $\mathbb{R}$. **R** expands $\mathbb{R}$, but has Dedekind-gaps of its own. However, notions of gaps are abundant, including set-theoretic (Dedekind), algebraic, etc. Whether a number system is gap-less depends on which structure it is viewed from. From **R**'s point of view, $\mathbb{R}$ is full of holes, it is *nowhere* gap-free: at each $r \in \mathbb{R}$ there is an infinite monad surrounding it which is missing from $\mathbb{R}$ ([31]:26).[16] We can 'fill-in' between two numbers in $\mathbb{R}$. But this is just to say that there is an order-type which injects into $\mathbb{R}$. The supporter of $\mathbb{R}$ must say that these gaps are somehow not relevant, that only Dedekind-gaps are significant for the essentially gap-free nature of a continuum. But, there are now reasons to think that Dedekind-gap closure cannot be necessary for continuity. Could **R** respect the motivation for Dedekind completeness?

An earlier concern was that Dedekind closure is unlike other closure properties. However this criticism is normally raised of **R**, claiming that **R** isn't a closure of $\mathbb{R}$, whilst $\mathbb{R} = \overline{\mathbb{Q}}^{Dedekind}$. But this is just false. With Tao, one can view **R** as the *elementary closure* of $\mathbb{R}$, where sequences of sentences which are "*potentially* simultaneously satisfiable" should be "*actually* simultaneously satisifiable" [61]:

Definition 33 ([61]:1)**.** $(\vec{x}_n)$ *is* elementarily Cauchy *if, for every predicate* $P(\vec{y})$*, the truth value of* $P(\vec{x}_n)$ *becomes eventually constant, writing this eventual truth value as* $\lim_{n\to\infty} P(\vec{x}_n)$*.*

$(\vec{x}_n)$ *is* elementarily convergent *to* $\vec{x}$ *if* $\lim_{n\to\infty} P(\vec{x}_n) = P(\vec{x})$ *for every predicate* $P(\vec{y})$*.*

For example, $(\pi + \frac{1}{n})$ converges elementarily in $\mathbb{R}$ in the language of ordered fields, but $(\frac{1}{n})$ is not elementary convergent, as each $\frac{1}{n}$ is distinguished from 0 by the predicate $P(x) := x = 0$ [61]. So, elementary completions can vary by language. There are also recognisable theorems, e.g. a Heine-Borel theorem analogue, and:

Theorem 34 (Bolzano-Weierstrass for Ultrapowers [61]:8)**.** *In an ultrapower* $\mathcal{U}$*, every sequence* (x_n) *of non-standard objects in* $\mathcal{U}$ *has an elementarily convergent subsequence* (x_{n_j})*.*

Definition 35 (Elementary Completeness [61]:7)**.** *A structure is* elementarily complete *if every elementarily Cauchy sequence* $(\vec{x}_n)$ *in it is elementarily convergent.*

[16] Its (**R**-)subspace is topologically discrete, "suggesting granularity even more" (Bankston, [2]).

$\mathbf{R}$ elementarily completes $\mathbb{R}$ ([61]:7). Elementary completeness is also a natural property. However, this has another name: (ω_1-)saturation. Still, with suitable framing, $\mathbf{R}$ completes $\mathbb{R}$. The question is whether this captures *continuity*.

Elementary completeness resembles Dedekind completeness somewhat, and even shares the same motivation: continuity seemingly means gaplessness. Dedekind-completeness is closure under gaps marked by any well-behaved (bounded) sets, even when these sets are undefinable, whilst elementary completeness is closure under gaps defined by well-behaved *properties*. Rather than checking that a bounded *set* of numbers has a least upper bound, we check the *properties* of sequences of numbers. So, elementary completeness is less directly connected to the ambient set theory. This might suggest that it supersedes Dedekind completeness as a condition for continuity.

For an honest comparison, elementary completeness should be subjected to the criticisms raised of Dedekind completeness. Both are not first-orderisable, so if this is conclusive for candidates for continuity, then both must be abandoned. This aside, elementary completeness does not imply the Archimedean property, and seems to do better in terms of measurement: it allows *any* kind of measurement (i.e. using any property), rather than simply the ordering on $\mathbb{R}$. Axiomatically assuming elementary closure is also a *minimal* extremal axiom, which is perhaps more expected for a closure axiom.

The significant worry is that elementary completion seems arbitrary, perhaps unlike Dedekind completeness, because of language dependency. One might claim that continua should be *homogeneous*, in the sense that linear translations should not affect their properties. But in $\mathbf{R}$, some Dedekind cuts are different from others, as not all have a least upper bound. For example, consider the language of ordered fields, and the map $x \mapsto x - \pi$. This changes which sequences are elementarily convergent. In the new field, the sequence $(\frac{1}{n})$ is elementary convergent, but $(\pi + \frac{1}{n})$ is not, the opposite of the original field. So defining continuity as elementary completeness makes properties of points in the continuum depend on the background language. This seems prima facie wrong.

The response is that this language is not rich enough, and should include many relations besides $<$. However, this doesn't explain the core problem. Certainly, the language of analysis should be rich, including the constants and relations which occur naturally in analysis.[17] But this dependency suggests that there is a *correct choice* of language for describing the continuum. Again, analysis might prompt a natural choice of language, so this may be unproblematic. But if so, this suggests that the continuum has a correct coordinate structure (deciding where 0 is, etc.). This is again puzzling if continua are homogeneous in the above sense.

The solution is to distinguish the geometric line and arithmetic continuum. This argument specifies a correct coordinate structure on the *arithmetic continuum*. But this is meant to be a number structure, so it should have a true 0 etc. (just as $\mathbb{N}$ has a true 0). Unlike the arithmetic continuum, the geometric line *does not* have an essential coordinate structure, order, or direction. This leaves an interesting open question about coordinatising the geometric line, but the arbitrary choice of a 0 on the geometric line does not seem to affect the arithmetic continuum.

The standardist might counter that although there is no order structure on the geometric line, we can extract a 'betweenness' relation on it. Then, they might say, this betweenness should be isomorphic to that induced by the order relation on the arithmetic continuum. Finally, they might argue that *this* relation should be homogeneous, but that, for the reasons mentioned, the between-relation of $\mathbf{R}$ is not homogeneous.

This non-homogeneity may be the pill the supporter of NSA must swallow. One possible response is that the standardist's notion of homogeneity is either vague or too set-theoretic. If homogeneity is kept vague, as 'a number in the arithmetic continuum is like any other', then it's false: the points should have different properties (e.g. positivity): so some *kinds* of properties must be chosen. So we again ask *which* properties are relevant. The standardist must reply that order properties demarked by sets are relevant, *even when those sets are not definable*. Meanwhile, elementary completeness is a way of closing any gap that *is* definable (or at least 'visible') to analysis. This is not merely a constructivist thought, it seems instead that Dedekind-gaps have more to do with *set theory* than the theory of continua. This seems to combat the claim that Dedekind completeness is less arbitrary than elementary completeness.

In general, elementary completeness seems a reasonable requirement for continuity. It looks like the 'gapless' story could be spun for both

[17] The naturality constraint stops elementary closure trivially becoming Dedekind closure.

Dedekind and elementary completeness. It may not be sufficient for continuity, but can be read as a necessary condition which helps explain why $\mathbf{R}$ is a good candidate for the continuum.

8 Survey

The standard account of the arithmetic continuum is that it is uniquely captured by $\mathbb{R}$. An alternative, $\mathbf{R}$, enriches $\mathbb{R}$ by including infinitely small and (some) infinite numbers. $\mathbf{R}$ better captures certain continuous phenomena, fits well with student intuitions about analysis, and has mathematical applications for which there is no standard interpretation. Some arguments for $\mathbf{R}$ superficially seem to prioritise $\mathbb{R}$, though this may be because $\mathbb{R}$ is a well-behaved subsystem. There are even reasons to think that it may be a better candidate that $\mathbb{R}$ at capturing the intuitive notion of continuity. Accounts with favour $\mathbb{R}$ rely on a particular notion of 'gap-free' which, amongst other problems, excludes otherwise gap-free structures, may be circular, and is closely tied up with the background set theory. An alternative notion of gap-free, which favours $\mathbf{R}$, weathers these criticisms, though somewhat questionably requires a fixed language for analysis. Overall, $\mathbb{R}$'s claim to the continuous throne is not uniquely strong. $\mathbf{R}$ wins out in several important metrics over $\mathbb{R}$, particularly mathematical use, so there are compelling reasons to think $\mathbf{R}$ might (also) be the (or a) continuum. The arithmetic continuum may contain very small numbers, but don't let that be frightening.

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