Mathematicians' Perceptions of "Generalisations" in Mathematics

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Why care?

We should care as generalisations occur all over mathematics:

- Theories: finite arithmetic to cardinal arithmetic
- Objects: $\mathbb{N}$ to cardinals, $\mathbb{R}$ to $\mathbb{R}_\kappa$, analytic to arbitrary functions

But! What do we know about them?

Plan for the Talk

Data: two small studies on mathematicians' use of generalisation

1. corpus study
2. interviews

Theory: the start of a theory of generalisations. End goal: an account of

1. distinctive features
2. metrics for evaluation
3. internal structure

Contents

Point of the Study

Broad question: how is "`generali*`" used in print?

Hypothesis: papers often use "`generali*`" to contextualise

Key Details

Corpus: Proceedings of the AMS 2019 (468 papers).

For each paper, we:

- **recorded the number of 'hits' of the text string** `generali*`. This picks up word forms containing this string, e.g. `generalised`, etc.
- **calculated a "processed" value,** P**,** to remove junk (e.g. hits in page titles, acknowledgements)

	Processed
Total	840
Mean	1.79
Median	1
Mode	0
Standard Deviation	4.48

Table: Hits-per-paper statistics

Evidence that "`generali*`" is used to contextualise

First, exclude 0s

Broad evidence: most papers have only 1 or 2 counts (low modal P)

Specific evidence: (linear regression test) 4.4% chance of no negative linear relationship between page length and hits per page

- **Conclusion:** there *is* a negative linear relationship, i.e. longer paper have fewer hits-per-page

Limitations and Solutions

- **Limitation 1**: evidence for special rôle of `generali*`, not necessarily contextualisation
- **Partial Solution**: new text analysis methods: word embedding model, topic modelling, coincidental nouns
- **Limitation 2**: uses 'self-reporting' - i.e. some intuitive examples of generalisations not so-called, possible inclusion of non-examples

Contents

Interviews

- Exploratory ethnographic study, professionals from various subfields - no logicians
- They were asked to **list examples** of good, bad, fruitful, successful generalisations.
- Suggest subfields are **systematic similar**/homogeneous in use of "generalisation" & measures of success
- **Responses fit with pre-experiment expectations**
- **Limitation:** small sample size, not representative in sociological sense

Contents

What Generalisations are Not

Other meta-theoretic processes. Argue:

1. Distinct extension from gen'n
2. Distinct intension from gen'n

Examples:

1. Easy: Scientific Gen'n - gen'n with **weak background assumptions**?
2. Hard: Abstraction - gen'n in class with **constant level of abstraction**? E.g. geometrical/analytic to arbitrary functions
3. also: domain expansion, concept expansion, idealisation, etc.

What Generalisations Are (Internal Structure and Kinds)

Classify generalisations in (at least) 2 dimensions:

1. **Ontology**: objects, theorems, proofs, structures, concepts, ...
   - 'everyday' ontology of mathematics?
2. **Exactness:** exact vs. analogous

Not just syntax! (Contra, e.g., Propp)

1. formal or natural language syntax? Canonical form and similarity relation **underdetermined**
2. No "recipe" for analogous generalisation - **unpredictable analogies**
3. What is *the* defining property? **Contingent properties** and **structure relativity**
4. Thoughts welcome!

Evaluation of Generalisations

1. (fruitful) generalisation is limited: **no 'generalisation without end'** - contra Kronecker, Chasles, Leibniz
2. Generalised proofs need **substantially new ideas** for success
3. Kitcher (Cournot, Mancosu?): **generalisations are fruitful if explanatory**
   - Not full story?
   - More abstract mathematics **not** explanatory, e.g. Cardinal arithmetic does not *explain* finite arithmetic (though it may unify)
   - Object generalisation - objects not explanatory alone!
   - Thoughts welcome!

Thank you!

References

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[2] Ellis A. B. *A Taxonomy for Categorizing Generalizations*, Journal of the Learning Sciences, 16(2), 2007.

[3] Kitcher P. *The nature of mathematical knowledge*, OUP, 1984.

[4] Mancosu P. *The philosophy of mathematical practice* OUP, 2008.

[5] Mancosu P. *Abstraction and Infinity*, OUP, 2016.

[6] Propp J. *Real Analysis in Reverse*, Am. Math. Mon. 120, 2013.

[7] Strachota S. *Conceptualizing generalization*, Open Mathematical Education Notes, vol. 6, 2016.

Not in Talk: Corpus Study Data

We recorded 7 values for each of the paper.

A length of the paper,

B frequency of `generali*` anywhere in the text,

C frequency of `without loss of generality` anywhere in the text,

D frequency of `generali*` in the title,

E frequency of `generali*` in the references,

F frequency of `generali*` in page headers,

G frequency of `generali*` in the acknowledgements.

Not in Talk: Corpus Study Graph

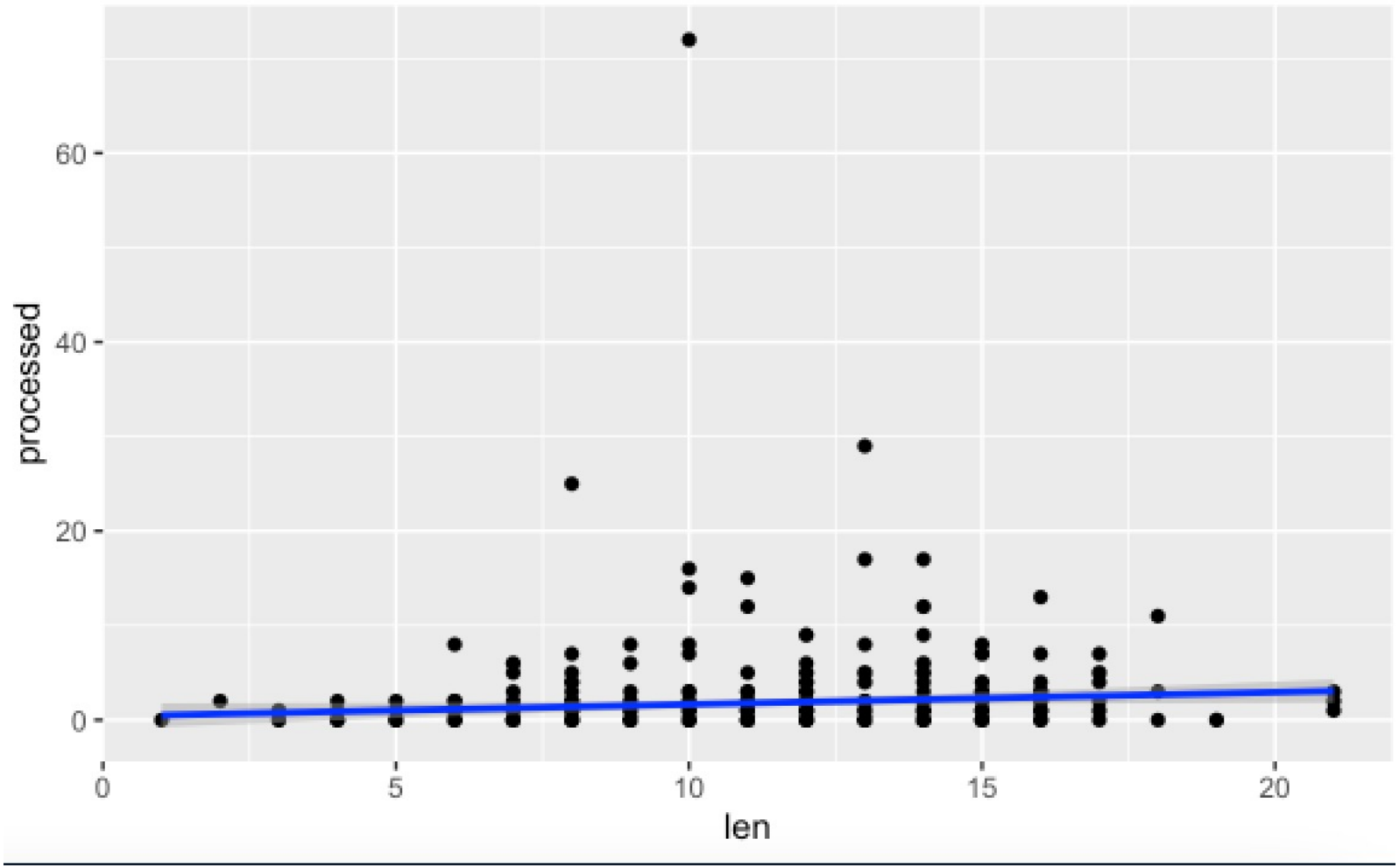

Figure: Graph: Length in pages (len) against Processed value (P)

Not in Talk: Corpus Study Statistics

Non-integer results are rounded to two decimal places.

	length (A)	generali (B)	wlog (C)	ref (D)	title (E)	header (F)	acknw (G)	processed (P)
Total	5185	1208	198	149	5	19	2	840
Mean	11.08	2.58	1.52	1.35	0.45	1.90	1.79	1.79
Median	11	1	1	1	0	0	1	1
Mode	11	0	1	1	0	0	1	0
SD	3.44	4.90	1.03	0.77	0.52	2.77	0.00	4.48

Table: total, mean, median, mode, and standard deviation of the data

Overall B/C ratio	6.10
Average of the individual B/C ratios	2.78

Table: `general/without loss of generality` ratios

Not in Talk: Corpus Study Observations

On data:

- More "`generali*`" occurred than initially speculated
- Fewer "`without loss of generality`" than initially speculated
- Outliers, $P > 10$, involved *names*, e.g. "Generalized torsion and decomposition of 3–manifolds"

Further limitation:

1. **Limitation 1**: no data on *where* `generali*` occurs in papers
2. **Solution**: richer corpus data
3. **Limitation 2**: published papers not representative of practice
4. **Partial Solution**: Informal corpora: "the great unread" e.g. FOM, Mathoverflow?
   1. FOM and HO/GM/LO in Mathoverflow are still not representative: somehow *about* maths not *within* maths
5. **Limitation 3** We miss variant phrases without `generali*`, e.g. "X is a more general version of Y" and "X, or more generally Y".

Not in Talk: More on Interviews

- No mention of ‘false’ generalisation
- The responses suggest “generalisation” is a **kind** in mathematics, not a façon de parler

Not in Talk: Distinct Metatheoretic Processes

- Concept expansion - only similar to **object gen'n**?
- Idealisation - some idealisations **add information**
- Domain expansion - **distinct domain and results** in gen'n of analysis to higher reals

Not in Talk: What Generalisations Are

- **Maths education view** (Ellis, Strachota) of mental processes only about *methods* - so only suitable for *proof* gen'n?