Views from a peak

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Amsterdam

A dilemma

1. metaphors and pictures

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2. unhelpful details

unhelpful details

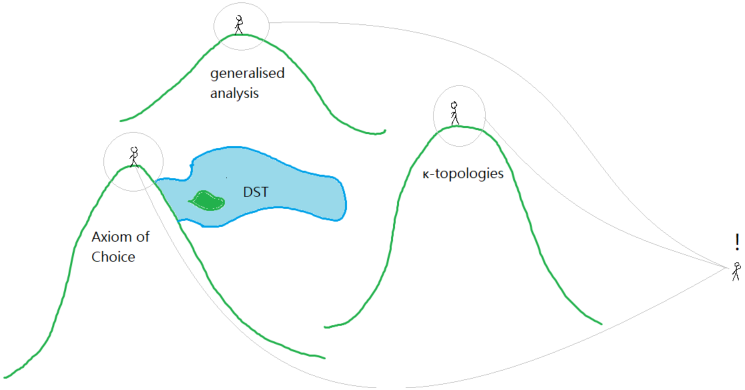
Lemma 3.3.33. For every $n > 2$, there is a $\Delta^{\mathbf{HC}}_{n-1}$, n-complete storage sequence from $\mathcal{M}$ in L .

Proof. Let $n > 2$. By Lemma 2.1.20, let $\Gamma \subset \omega_1 \times \mathbf{HC}$ be a universal $\Sigma^{\mathbf{HC}}_{n-2}$ set. We define the required sequence recursively. Let (M_0, P_0) be the $<_L$-least pair such that $(M_0, P_0) \in \mathcal{M}$. Suppose that $((M_{\xi'}, P_{\xi'}))_{\xi' \in \xi}$ is already defined. If ξ is a limit, then we set $P_\xi := \bigcup_{\xi' \in \xi} P_{\xi'}$ and let M_ξ be the $<_L$-least ctm of ZFC such that $(M_\xi, P_\xi) \in \mathcal{M}$ and M_ξ contains $((M_{\xi'}, P_{\xi'}))_{\xi' \in \xi}$. If $\xi = \xi' + 1$ is a successor, let (M_ξ, P_ξ) be the $<_L$-least pair such that:

1. $(M_{\xi'}, P_{\xi'})$ is strictly-$\preccurlyeq$ (M_ξ, P_ξ), and
2. either $(M_\xi, P_\xi) \in D_{\xi'} := \{m \in \mathcal{M} : (\xi', m) \in \Gamma\}$ or there is no $(N, Q) \in D_{\xi'}$ extending (M_ξ, P_ξ).

Recall that $<_L \restriction \mathbf{HC}^2$ is $\Delta^{\mathbf{HC}}_1$ (Lemma 2.1.19). By definition, $\mathcal{M}$ and $\preccurlyeq$ are $\Delta^{\mathbf{HC}}_1$, and Γ is $\Sigma^{\mathbf{HC}}_{n-2}$, so $((M_\xi, P_\xi))_{\xi \in \omega_1}$ is $\Delta^{\mathbf{HC}}_{n-1}$. By Property 2., the sequence is n-complete. $\square$

a picture



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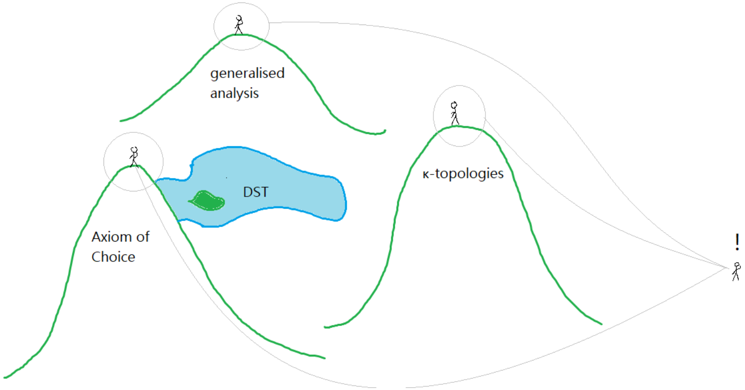

- the lake: descriptive set theory (DST)

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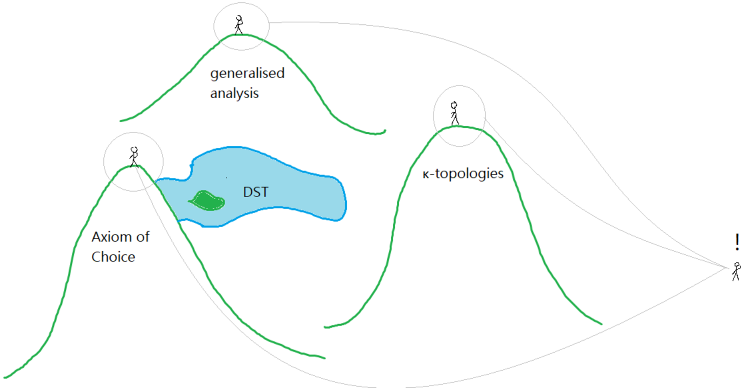

- the lake: descriptive set theory (DST)
- three mountains: Axiom of Choice, generalised DST, and κ-topologies

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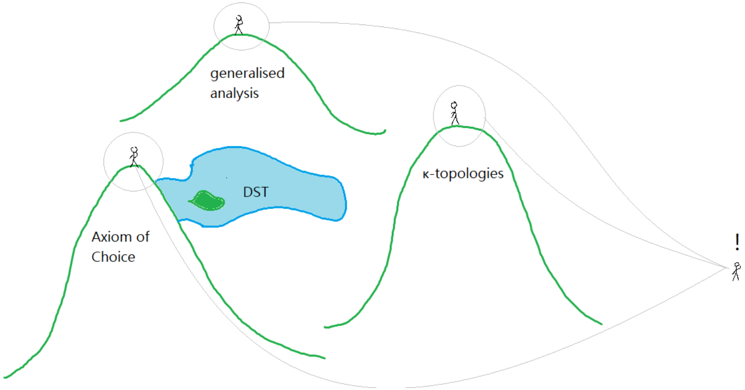

- the lake: descriptive set theory (DST)
- three mountains: Axiom of Choice, generalised DST, and κ-topologies
- the moral of the story: putting the evidence together to find a philosophical account of generalisations in mathematics

(Descriptive) Set Theory

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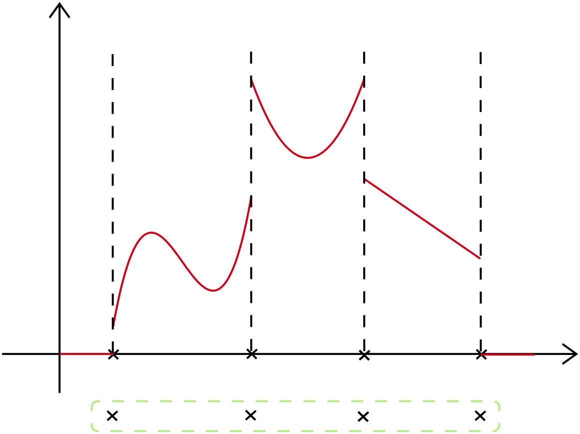
- DST is (mainly) about sets of (real) numbers which have a 'nice description'
- this is linked closely with graphs (and 'analysis', the continuation of secondary school calculus)

picture of set of discontinuities



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  2. possibly by making something more *inclusive*
  3. ...

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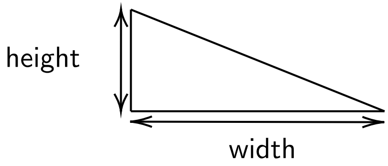

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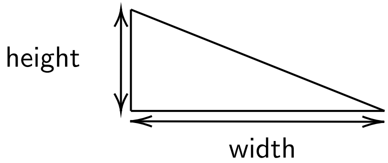
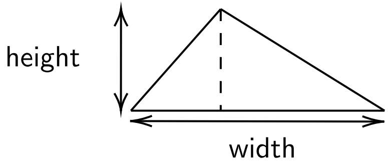


- generalised case: area of **any** triangle equals $1/2$ height $\times$ width



three mountains: generalising DST

- Normally, the natural numbers $(1, 2, 3, ...)$ form a 'backbone' of the number line, $\mathbb{R}$.
- We have a longer number line, which has a larger infinity as the 'backbone', $\mathbb{R}_\kappa$, which has numbers like ∞, $\frac{1}{\infty}$, and so on

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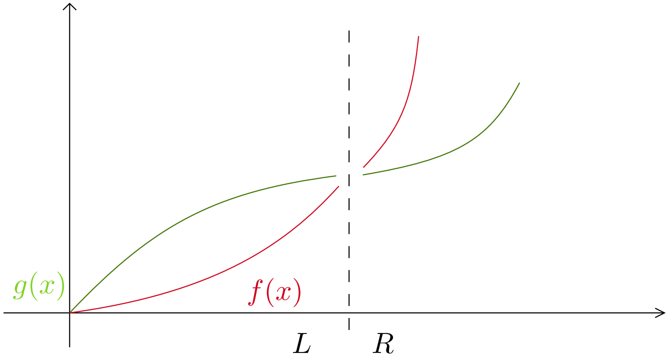
- Graphs can be strange, e.g. crossing without intersecting:



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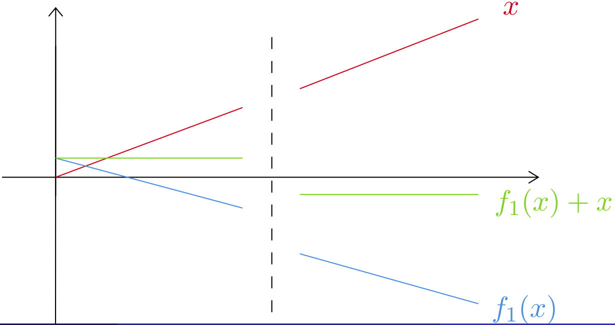

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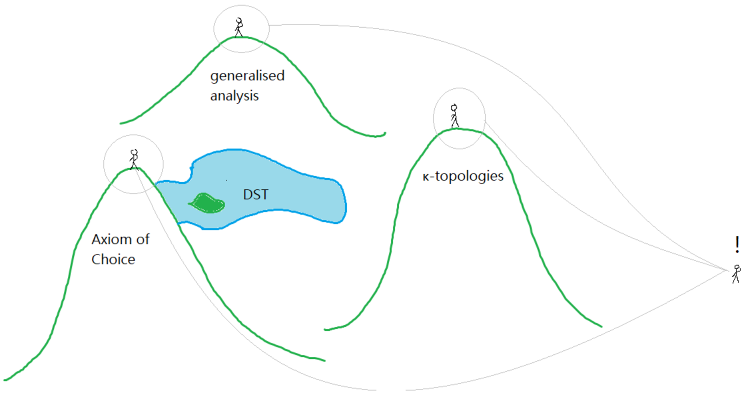
- some things are no longer true for $\mathbb{R}_\kappa$, e.g. adding κ-continuous need not be κ-continuous:
- yet other things depend on the size of infinity (κ has tree property iff sharp functions have extreme points)

last lap: back to the philosophy



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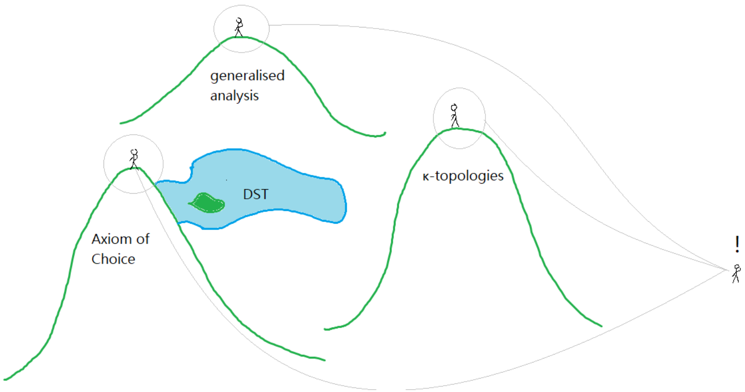

Our first go at describing mathematical generalisations: make a theorem/proof/approach more *abstract*, more *inclusive*, ...

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Last chapter gives a philosophical account of generalisation. Three main points:

1. Sui generis: generalisation *not just* abstraction or expansion
2. Motivations: more than *explanatoriness* (in fact, some generalisations are bad at explaining)
3. Technique: generalisation not mechanical. I.e. not syntactic process e.g. not just replacing constants by variables

Thank you!