Views from a peak

Ned Wontner (ILLC, UvA)

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Amsterdam
A dilemma

- metaphors and pictures
A dilemma

1. metaphors and pictures
2. unhelpful details
Lemma 3.3.33. For every $n > 2$, there is a $\Delta^\text{HC}_{n-1}$, $n$-complete storage sequence from $\mathcal{M}$ in $L$.

Proof. Let $n > 2$. By Lemma 2.1.20, let $\Gamma \subset \omega_1 \times \text{HC}$ be a universal $\Sigma^\text{HC}_{n-2}$ set. We define the required sequence recursively. Let $(M_0, P_0)$ be the $<_L$-least pair such that $(M_0, P_0) \in \mathcal{M}$. Suppose that $((M_{\xi'}, P_{\xi'}))_{\xi' \in \xi}$ is already defined. If $\xi$ is a limit, then we set $P_\xi := \bigcup_{\xi' \in \xi} P_{\xi'}$ and let $M_\xi$ be the $<_L$-least ctm of $\text{ZFC}$ such that $(M_\xi, P_\xi) \in \mathcal{M}$ and $M_\xi$ contains $((M_{\xi'}, P_{\xi'}))_{\xi' \in \xi}$. If $\xi = \xi' + 1$ is a successor, let $(M_\xi, P_\xi)$ be the $<_L$-least pair such that:

1. $(M_{\xi'}, P_{\xi'})$ is strictly-$\prec_\Gamma (M_\xi, P_\xi)$, and

2. either $(M_\xi, P_\xi) \in D_{\xi'} := \{ m \in \mathcal{M} : (\xi', m) \in \Gamma \}$ or there is no $(N, Q) \in D_{\xi'}$ extending $(M_\xi, P_\xi)$.

Recall that $<_L\mid\text{HC}^2$ is $\Delta^\text{HC}_1$ (Lemma 2.1.19). By definition, $\mathcal{M}$ and $\prec_\Gamma$ are $\Delta^\text{HC}_1$, and $\Gamma$ is $\Sigma^\text{HC}_{n-2}$, so $((M_\xi, P_\xi))_{\xi \in \omega_1}$ is $\Delta^\text{HC}_{n-1}$. By Property 2., the sequence is $n$-complete. \qed
The lake: descriptive set theory (DST)

Three mountains: Axiom of Choice, generalised DST, and $\kappa$-topologies

The moral of the story: putting the evidence together to find a philosophical account of generalisations in mathematics.
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• three mountains: Axiom of Choice, generalised DST, and $\kappa$-topologies
• the moral of the story: putting the evidence together to find a philosophical account of generalisations in mathematics
(Descriptive) Set Theory

- set theory is about sets
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- we can study the infinite, e.g. the set of all whole numbers, various subsets of this
- also has a ‘foundational’ rôle
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DST is (mainly) about sets of (real) numbers which have a ‘nice description’
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DST is (mainly) about sets of (real) numbers which have a ‘nice description’

this is linked closely with graphs (and ‘analysis’, the continuation of secondary school calculus)
picture of set of discontinuities
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(often?) means to ‘improve’ a theorem/proof/approach

1. possibly by making something more abstract
2. possibly by making something more inclusive
3. ...

first go at generalisation in mathematics
Generalisation is made of a base case and a generalised case:
Warm up: generalisation to find the area of a triangle

Generalisation is made of a base case and a generalised case:

- **Base case**: Area of a right-angle triangle equals \( \frac{1}{2} \) height \( \times \) width.

![Diagram of a right-angle triangle with height and width labels.]

- **Generalised case**: Area of any triangle equals \( \frac{1}{2} \) height \( \times \) width.
warm up: generalisation to find the area of a triangle

Generalisation is made of a base case and a generalised case:

- **base case**: area of a **right-angle** triangle equals $\frac{1}{2} \text{height} \times \text{width}$

  ![Right-angle triangle diagram]

- **generalised case**: area of **any** triangle equals $\frac{1}{2} \text{height} \times \text{width}$

  ![Any triangle diagram]
Normally, the natural numbers \( (1, 2, 3, \ldots) \) form a ‘backbone’ of the number line, \( \mathbb{R} \).

We have a longer number line, which has a larger infinity as the ‘backbone’, \( \mathbb{R}_\kappa \), which has numbers like \( \infty, \frac{1}{\infty} \), and so on.
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I looked at properties of graphs on $\mathbb{R}_\kappa$.

Graphs can be strange, e.g. crossing without intersecting:

![Graph Diagram]

\[ g(x) \]

\[ f(x) \]

$\mathbb{R}$
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Specifically:

- some things generalise to \(\mathbb{R}_\kappa\) (e.g. intermediate value theorem)
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Specifically:

- some things generalise to \(\mathbb{R}_\kappa\) (e.g. intermediate value theorem)
- some things are no longer true for \(\mathbb{R}_\kappa\), e.g. adding \(\kappa\)-continuous need not be \(\kappa\)-continuous:
- yet other things depend on the size of infinity (\(\kappa\) has tree property iff sharp functions have extreme points)
last lap: back to the philosophy

Our first go at describing mathematical generalisations: make a theorem/proof/approach more abstract, more inclusive, ...

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Views from a peak: Generalisations and Descriptive Set Theory
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Our first go at describing mathematical generalisations: make a theorem/proof/approach more \textit{abstract}, more \textit{inclusive}, ...
What are generalisations anyway

Last chapter gives a philosophical account of generalisation. Three main points:

1. Sui generis: generalisation not just abstraction or expansion
2. Motivations: more than explanatoriness (in fact, some generalisations are bad at explaining)
3. Technique: generalisation not mechanical. I.e. not syntactic process
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3. Technique: generalisation not mechanical. I.e. *not* syntactic process e.g. *not just* replacing constants by variables
Thank you!