Views from a peak

Ned Wontner (ILLC, UvA)

25th September 2023
Amsterdam
metaphors and pictures
A dilemma

1. metaphors and pictures
2. unhelpful details
Lemma 3.3.33. For every $n > 2$, there is a $\Delta_{n-1}^\text{HC}$, $n$-complete storage sequence from $\mathcal{M}$ in $L$.

Proof. Let $n > 2$. By Lemma 2.1.20, let $\Gamma \subset \omega_1 \times \text{HC}$ be a universal $\Sigma_{n-2}^\text{HC}$ set. We define the required sequence recursively. Let $(M_0, P_0)$ be the $<_L$-least pair such that $(M_0, P_0) \in \mathcal{M}$. Suppose that $((M_{\xi'}, P_{\xi'}))_{\xi' < \xi}$ is already defined. If $\xi$ is a limit, then we set $P_\xi := \bigcup_{\xi' < \xi} P_{\xi'}$ and let $M_\xi$ be the $<_L$-least ctm of $\text{ZFC}$ such that $(M_\xi, P_\xi) \in \mathcal{M}$ and $M_\xi$ contains $((M_{\xi'}, P_{\xi'}))_{\xi' < \xi}$. If $\xi = \xi' + 1$ is a successor, let $(M_\xi, P_\xi)$ be the $<_L$-least pair such that:

1. $(M_{\xi'}, P_{\xi'})$ is strictly-$\preceq$ $(M_\xi, P_\xi)$, and
2. either $(M_\xi, P_\xi) \in D_{\xi'} := \{m \in \mathcal{M} : (\xi', m) \in \Gamma\}$ or there is no $(N, Q) \in D_{\xi'}$ extending $(M_\xi, P_\xi)$.

Recall that $<_L\upharpoonright\text{HC}^2$ is $\Delta_1^\text{HC}$ (Lemma 2.1.19). By definition, $\mathcal{M}$ and $\preceq$ are $\Delta_1^\text{HC}$, and $\Gamma$ is $\Sigma^\text{HC}_{n-2}$, so $((M_\xi, P_\xi))_{\xi \in \omega_1}$ is $\Delta_{n-1}^\text{HC}$. By Property 2, the sequence is $n$-complete.
a picture

Ned Wontner
the lake: descriptive set theory (DST)
the lake: descriptive set theory (DST)
three mountains: Axiom of Choice, generalised DST, and \( \kappa \)-topologies
- the lake: descriptive set theory (DST)
- three mountains: Axiom of Choice, generalised DST, and \( \kappa \)-topologies
- the moral of the story: putting the evidence together to find a philosophical account of generalisations in mathematics
(Descriptive) Set Theory

- set theory is about, well, sets
(Descriptive) Set Theory

- set theory is about, well, sets
  - we can study the infinite (like astronomy in physics?), e.g. the set of all whole numbers, various subsets of this
  - also has a ‘foundational’ rôle (like atoms in physics)
set theory is about, well, sets
- we can study the infinite (like astronomy in physics(?)), e.g. the set of all whole numbers, various subsets of this
- also has a ‘foundational’ rôle (like atoms in physics)

DST is (mainly) about sets of (real) numbers which have a ‘nice description’
set theory is about, well, sets
- we can study the infinite (like astronomy in physics(?)), e.g. the set of all whole numbers, various subsets of this
- also has a ‘foundational’ rôle (like atoms in physics)

DST is (mainly) about sets of (real) numbers which have a ‘nice description’

this is linked closely with graphs (and ‘analysis’, the continuation of secondary school calculus)
picture of set of discontinuities
generalisation in mathematics is typically a good thing! (Maybe unlike other uses of the word “generalisation”)
generalisation in mathematics is typically a good thing! (Maybe unlike other uses of the word “generalisation”)

(often?) means to ‘improve’ a theorem/proof/approach

1. possibly by making something more abstract (removing details specific to one case/conceptually ‘higher’)
2. possibly by making something more inclusive (covers more cases)
3. ...

...
Generalisation is made of a base case and a generalised case:
Generalisation is made of a base case and a generalised case:

- base case: area of a **right-angle** triangle equals $\frac{1}{2}$ height $\times$ width

![Diagram of a right-angled triangle with height and width labels.]
warm up: generalisation to find the area of a triangle

Generalisation is made of a base case and a generalised case:

- base case: area of a **right-angle** triangle equals $\frac{1}{2} \text{height} \times \text{width}$

- generalised case: area of any triangle equals $\frac{1}{2} \text{height} \times \text{width}$
Normally, the natural numbers \((1, 2, 3, ...)\) form a ‘backbone’ of the number line, \(\mathbb{R}\).

We have a longer number line, which has a larger infinity as the ‘backbone’, \(\mathbb{R}_\kappa\), which has numbers like \(\infty, \frac{1}{\infty}\), and so on.
Normally, the natural numbers \((1, 2, 3, \ldots)\) form a ‘backbone’ of the number line, \(\mathbb{R}\).

We have a longer number line, which has a larger infinity as the ‘backbone’, \(\mathbb{R}_\kappa\), which has numbers like \(\infty, \frac{1}{\infty}\), and so on.

I looked at properties of graphs on \(\mathbb{R}_\kappa\).
Normally, the natural numbers \((1, 2, 3, \ldots)\) form a ‘backbone’ of the number line, \(\mathbb{R}\).

We have a longer number line, which has a larger infinity as the ‘backbone’, \(\mathbb{R}_\kappa\), which has numbers like \(\infty, \frac{1}{\infty}\), and so on.

I looked at properties of graphs on \(\mathbb{R}_\kappa\).

Graphs can be strange, e.g. crossing without intersecting:

\[
g(x) \\
\begin{array}{c}
g(x) \quad f(x) \\
L \\
R
\end{array}
\]
Normally, the natural numbers (1, 2, 3, ...) form a ‘backbone’ of the number line, \( \mathbb{R} \).

We have a longer number line, which has a larger infinity as the ‘backbone’, \( \mathbb{R}_\kappa \), which has numbers like \( \infty, \frac{1}{\infty} \), and so on.

Specifically

- some things generalise to \( \mathbb{R}_\kappa \) (e.g. intermediate value theorem)
Normally, the natural numbers \((1, 2, 3, \ldots)\) form a ‘backbone’ of the number line, \(\mathbb{R}\).

We have a longer number line, which has a larger infinity as the ‘backbone’, \(\mathbb{R}_\kappa\), which has numbers like \(\infty, \frac{1}{\infty}\), and so on.

Specifically:

- Some things generalise to \(\mathbb{R}_\kappa\) (e.g. intermediate value theorem).
- Some things are no longer true for \(\mathbb{R}_\kappa\), e.g. adding \(\kappa\)-continuous need not be continuous.
Normally, the natural numbers \((1, 2, 3, \ldots)\) form a ‘backbone’ of the number line, \(\mathbb{R}\).

We have a longer number line, which has a larger infinity as the ‘backbone’, \(\mathbb{R}_\kappa\), which has numbers like \(\infty, \frac{1}{\infty}\), and so on.

Specifically:

- Some things generalise to \(\mathbb{R}_\kappa\) (e.g. intermediate value theorem).
- Some things are no longer true for \(\mathbb{R}_\kappa\), e.g. adding \(\kappa\)-continuous need not be \(\kappa\)-continuous:
- Yet other things depend on the size of infinity (\(\kappa\) has tree property iff sharp functions have extreme points).
last lap: back to the philosophy

Our first go at describing mathematical generalisations: make a theorem/proof/approach more abstract, more inclusive, ...

Axiom of Choice

generalised analysis

DST

κ-topologies
Our first go at describing mathematical generalisations: make a theorem/proof/approach more *abstract*, more *inclusive*, ...
Last chapter gives a philosophical account of generalisation. Three main points:
Last chapter gives a philosophical account of generalisation. Three main points:

1. Sui generis: generalisation *not just* abstraction or expansion
What are generalisations anyway

Last chapter gives a philosophical account of generalisation. Three main points:

1. Sui generis: generalisation *not just* abstraction or expansion
2. Motivations: more than *explanatoriness* (in fact, some generalisations are bad at explaining)
Last chapter gives a philosophical account of generalisation. Three main points:

1. Sui generis: generalisation *not just* abstraction or expansion

2. Motivations: more than *explanatoriness* (in fact, some generalisations are bad at explaining)

3. Technique: generalisation not mechanical. I.e. not syntactic process e.g. not just replacing constants by variables
Thank you!