Big $\mathbb{R}$: High Cardinality Fields and Their Analysis

Ned Wontner (ILLC, UvA)

1st September 2021
British Logic Colloquium 2021, Durham

- Generalise the real numbers ($\mathbb{R}$) to *larger sizes infinity*
- Axiomatic $\mathbb{F}(\kappa)$ and construction $\mathbb{R}_\kappa$
- Generalised real analysis
  - Bolzano-Weierstrass - cardinal properties
  - Intermediate value theorem - gaps and continuity
  - Stone-Weierstrass theorem - infinite sums?

Contents

Cardinals and Reals

Cardinals: 'sizes of infinity'

- '0th' cardinal is the natural numbers $\{0, 1, 2, 3...\} = \mathbb{N}$, also called $\aleph_0$
- the next one is $\aleph_1$, then $\aleph_2$ etc.
- for simplicity, assume $\kappa^{<\kappa} = \kappa$ (n.b. implies κ is regular and 'CH_{κ^-}')

$\mathbb{R}$: all the fractions ($\mathbb{Q}$) and all the infinite decimals

- $\mathbb{R}$ has order and field structure: we have $\times, +$ and $<$, i.e. we are considering $(\mathbb{R}, +, <, \times, 0)$
- not just ω^ω or $\mathcal{P}(\omega)$, as need field structure

Contents

Generalising $\mathbb{R}$: Axioms for $\mathbb{F}(\kappa)$

Some axioms:

1. Size: $|\,\mathbb{R}\,| = 2^{\aleph_0}$ so: $|\,\mathbb{F}(\kappa)| = 2^{\kappa}$
2. Weight: $\mathbb{Q}$ dense in $\mathbb{R}$ & $|\,\mathbb{Q}\,| = \aleph_0$ so: $\mathbb{Q}(\kappa)$ dense in $\mathbb{F}(\kappa)$ & $|\,\mathbb{Q}(\kappa)| = \kappa$
3. Cauchy: $\mathbb{R}$ is Cauchy complete (all (ω-)Cauchy sequence have limits) so: $\mathbb{F}(\kappa)$ (κ-)Cauchy complete
4. Algebra etc.: $\mathbb{R}$ is an ordered rcf (first order theory of $\mathbb{R}$), so: $\mathbb{F}(\kappa)$ is an ordered rcf

n.b.: also require $\mathbb{F}(\kappa)$ is η_κ, as $\mathbb{R}$ is $\eta_{\aleph_0}$

$\overline{\mathbb{Q}_\kappa}^{Cauchy}$, or $\mathbb{R}_\kappa$

There is such an ordered field!

Construct $\mathbb{Q}_\kappa$, then take its Cauchy closure

- $\mathbb{Q}_\kappa = \{0,1\}^{<\kappa}$
- Order is lexicographic, with $0 <$ 'undefined' < 1, so $0010 < 001 < 0011$

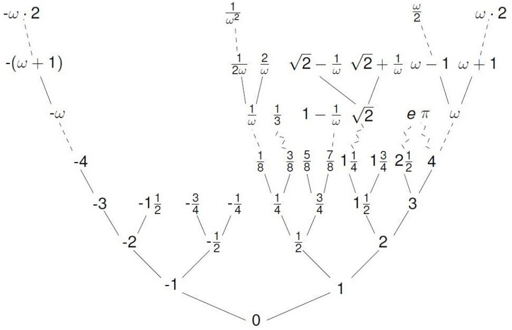


Figure: $\mathbb{Q}_\kappa$ tree (sequences renamed to be more familiar)

Th'm: $\mathbb{R}_\kappa$ is a field

- Surreal numbers: $No = \{0,1\}^{Ordinals}$ with same order
- $+, \times$ etc. defined inductively on No, so it is (class-sized) Field.
- $\mathbb{Q}_\kappa$ is an initial subset of $No \Rightarrow \mathbb{Q}_\kappa$ is a field
- $\Rightarrow$ Cauchy closure $\mathbb{R}_\kappa = \overline{\mathbb{Q}_\kappa}$ is a field

etc...

Trouble with gaps

- $\mathbb{R}_\kappa$ has 'gaps' (e.g. $\aleph_0, \kappa$)
- E.g. {Finite or negative $\mathbb{R}_\kappa$} $<$ {infinite, positive $\mathbb{R}_\kappa$} but no lub/glb

- Unavoidable! "$\mathbb{F}(\kappa)$ has no gaps" (Dedekind complete) $\Rightarrow \kappa = \aleph_0$, i.e. only $\mathbb{R}$ is gap-free
- Coping with the gaps:
  - Intuitive? *shouldn't be* a largest finite or smallest infinite number!
  - fill 'fillable' gaps: gaps approached by a Cauchy sequence *are* filled
- behaviour of functions at gaps?

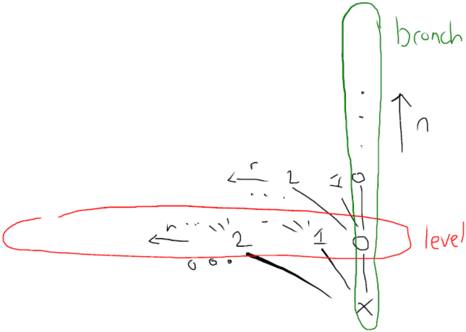
Contents

Some Theorems on $\mathbb{R}$

- Bolzano-Weierstrass (BWT): bounded sequences have convergent subsequences
- Intermediate Value (IVT): continuous functions go through all the intermediate points
- Stone-Weierstrass (SWT): every continuous function is (uniformly) approximated by polynomials

Bolzano-Weierstrass Theorem

- Th'm (CGHL [2]): weak-BWT(κ) iff κ has tree property.
- Tree property for (κ): κ-tree T with $|T| = \kappa$, then either some branch has length κ or some level has width κ.



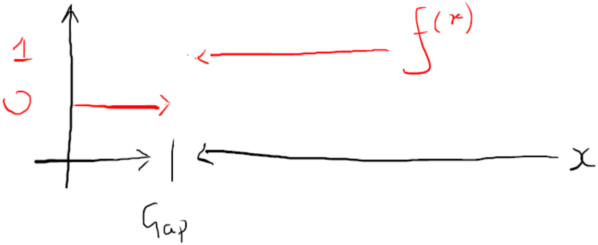
Diversion: Reverse Maths & Computable Analysis

- Th'm (Friedman, '75): Over RCA_0: '$\aleph_0$ has the tree property' $\equiv_{RM}$ BWT($\mathbb{R}$)
- Th'm (Brattka, Gherardi, Marcone '12): BWT($\mathbb{R}$) $\equiv_W$ *jump* of a weakening of '$\aleph_0$ has the tree property'[1]
- Other candidate $\equiv$ to look at for $\mathbb{R}_\kappa$?
- Open question: Extreme values for $\sim$continuous functions on $\mathbb{R}_\kappa$ iff tree property for κ?

[1] Weak Kőnig's lemma

Intermediate Value Theorem

- IVT($\mathbb{R}_\kappa$) fails: $\exists$ continuous functions which *jump at gaps*



- Solution: (κ-)topological, not sequential definition of continuity
- Th'm (Galeotti): "κ-continuous" $\Rightarrow$ IVT
- Th'm (W.): continuous + IVT + a bit more $\Rightarrow$ "κ continuous"
- Tighter restrictions on continuity imply more, e.g. Th'm (W.): "κ-*super* continuity" $\Rightarrow$ Rolle's theorem, etc...

Stone-Weierstrass

Any notion of continuity satisfy SWT?

- Problem: polynomials are too short! $X^5 + \omega X^4 + \frac{\sqrt{\omega}}{2}$
- Long polynomials:
  - infinite exponents: X^ω
  - infinitely many terms: $\sum_{\alpha<\beta} c_\alpha X^{i_\alpha}$

Large exponents easier: $exp(-)$ defined on $\mathbb{R}^+_\kappa$, so $X^\alpha := exp(\alpha.ln(X))$

Infinitely many terms hard: definition of infinite sum?

Jackpot: infinite sums

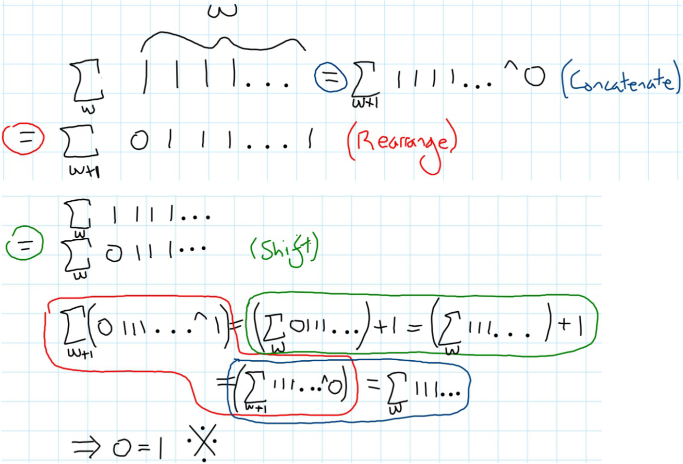
What is $\Sigma_\beta r_\alpha$ for $\beta \geq \omega$? ($r_\alpha \in \mathbb{R}_\kappa$)

Even countable $\mathbb{R}$-sums fail: $(\Sigma_{n \leq N} \frac{1}{2^n})_{N \in \omega}$ does not converge in $\mathbb{R}_\kappa$

- Model theoretic sum (à la non-standard analysis, [7]): let $f(n) = \Sigma_{\alpha \leq n} x_\alpha$ (unique). $\Sigma_\beta^{RSS}(x_\alpha) := f^*(\beta)$. Problem: does not depend on x_α values after ω
- $\Sigma^{FB} x_\alpha = sup\{$finite partial sums$\}$. Problem: no infinite sequence is summable
- etc...

Th'm (Galeotti, W.): There are no linear, rearrangable, finite shifting, concatenating definitions of sum which extend the (finite) $\mathbb{R}_\kappa$ algebra, sum a non-trivial infinite sequence and satisfy comparison

Proof idea:



Thank you!

References

[1] Brattka, V., Gherardi, G., Marcone, A. *The Bolzano–Weierstrass theorem is the jump of weak Kőnig's lemma*, APAL, 2012.

[2] Carl, M., Galeotti, L., Löwe, B. *The Bolzano-Weierstrass theorem in generalised analysis*.

[3] Conway, J. H., *On Numbers and Games*, AP, 1976.

[4] Friedman, H. *Some Systems of Second Order Arithmetic and Their Use*, Proc ICM, 1975.

[5] Galeotti, L. *Computable analysis over the generalized Baire space*, UvA, 2015.

[6] Galeotti, L. *The theory of the generalised real numbers and other topics in logic*, UHH, 2019.

[7] Rubinstein-Salzedo, S., Swaminathan, A. *Analysis on Surreal Numbers*, 2013.

Rolle's theorem

if f is continuous and there are $a \neq b$ such that $f(a) = f(b)$ then there's a point c where $f'(c) = 0$, i.e. stationary

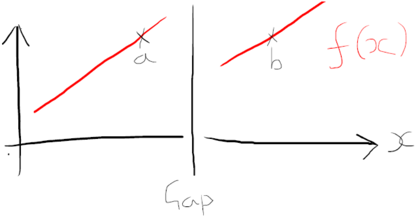


Figure: Bad function: continuous, non-Rolle's

- κ-*super* continuous: $\forall [a,b] (cof(f^{-1}[a,b]), coi(f^{-1}[a,b]) < \kappa)$
- Th'm (W.): κ-super continuous function f satisfy Rolle's
  Proof idea: f takes 'extreme values' on $[a,b]$, so local maxima can't be 'at' a gap. Then as in $\mathbb{R}$.
- Open question: Rolle's for κ-continuous?