Descriptive Set Theory via Analysis, and its Generalisation to Higher Cardinals

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Plan of Attack

Part 1

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- Classical descriptive set theory and real analysis incl. measure theory

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  - Baire functions
  - Borel sets
  - Analytic sets
  - regularity properties

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- Generalised descriptive set theory
- Generalised real analysis
  - A new field, $\mathbb{R}_\kappa$
  - Generalising theorems of real analysis

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Figure: Henri Lebesgue (1875-1941)

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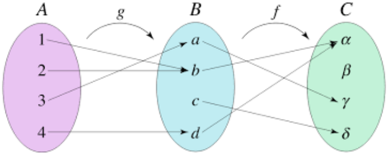

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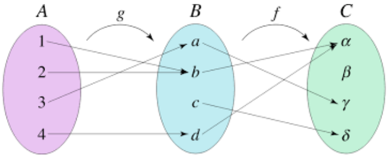

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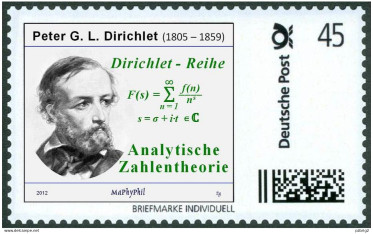

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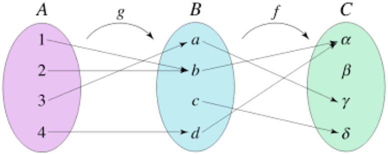

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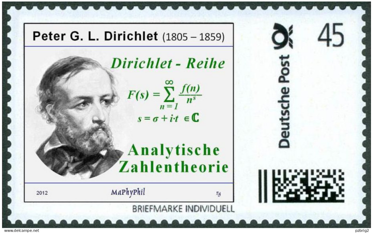

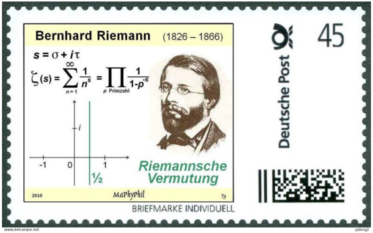

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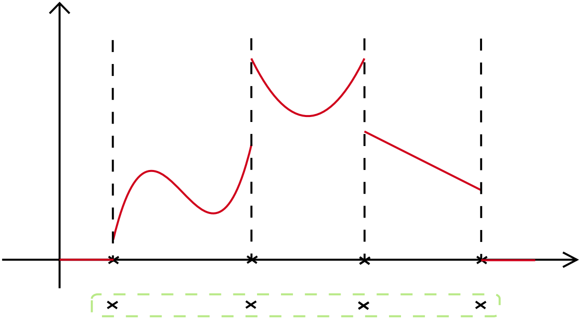

Figure: picture of a set of discontinuities

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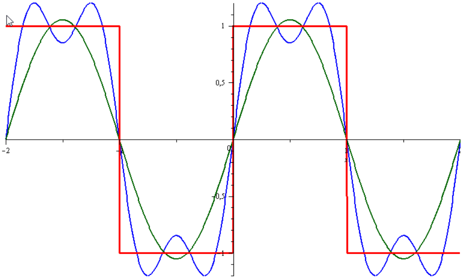

Figure: Fourier series of level 0 functions whose limit is level 1

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- notion and hierarchy of **definability** for (interesting) functions

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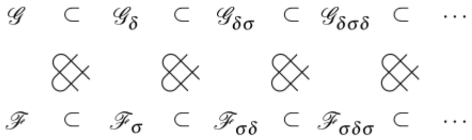

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- in fact THEOREM (Lebesgue, 1905): the Borel hierarchy IS the Baire function hierarchy (Baire $\alpha = \Sigma^0_{\alpha+1}$-measurable)

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- again, no! Still have nice **regularity** properties

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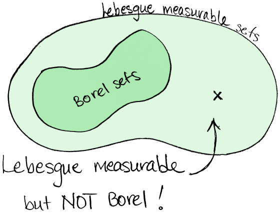
- but every Borel/analytic set is BP, PSP, LM - good for real analysis!



from math3ma.com/blog/lebesgue-but-not-borel

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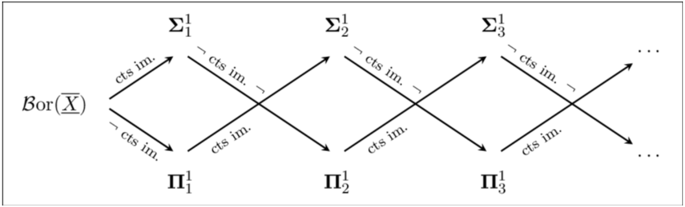

Figure: higher and higher above the Borels: the projective hierarchy

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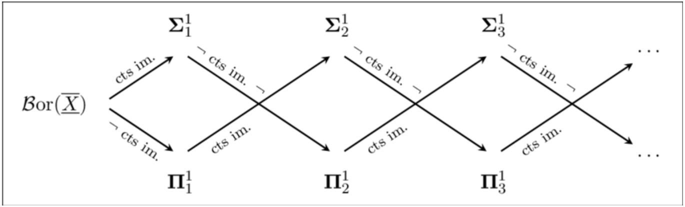

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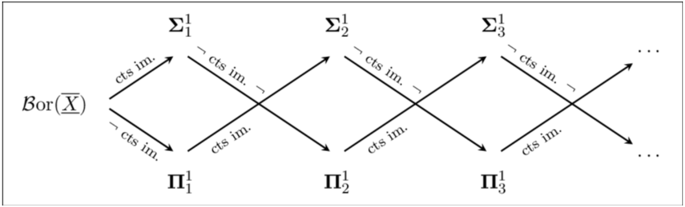

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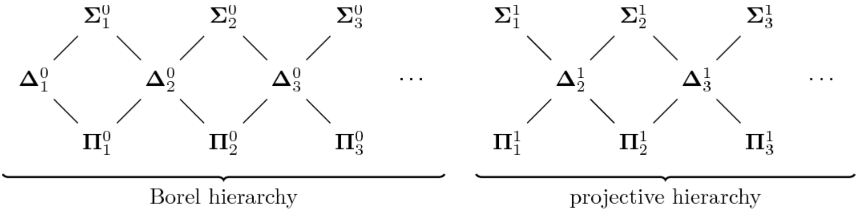
End of Part 1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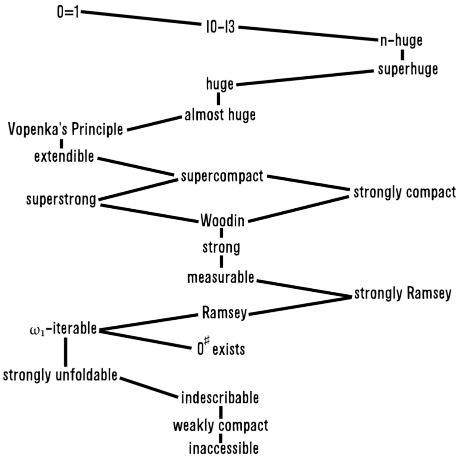

0=1
I0–I3
n-huge
superhuge
huge
almost huge
Vopenka's Principle
extendible
supercompact
strongly compact
superstrong
Woodin
strong
measurable
strongly Ramsey
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strongly unfoldable
indescribable
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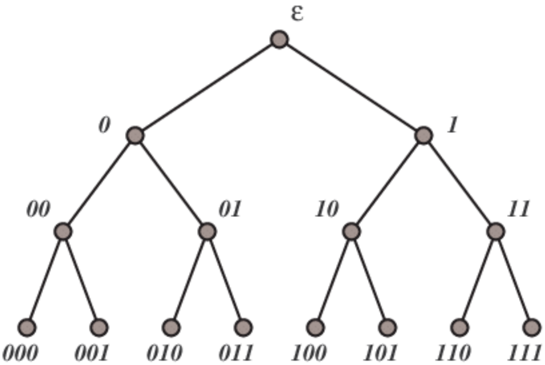

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- Amazingly, yes! (Galeotti, 2015) e.g. use Surreal numbers

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What is $\mathbb{R}_\kappa$ like?

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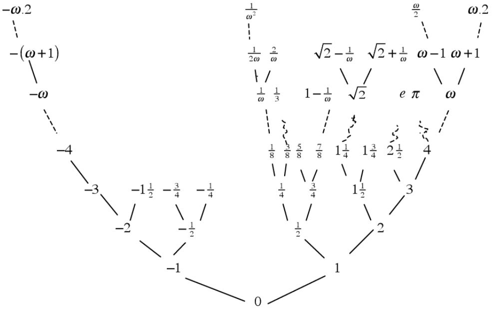

Figure: from Costin, Ehrlich, and Friedman 2015

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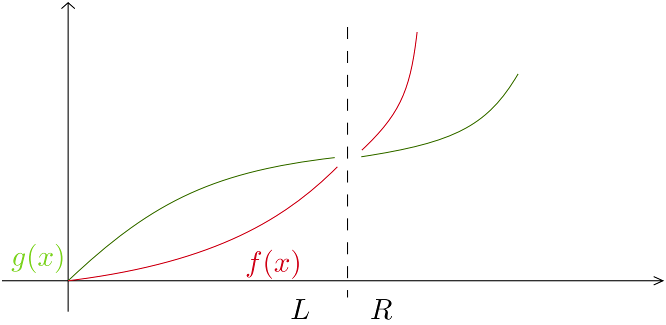
- gaps cause strange properties of $\mathbb{R}_\kappa$ functions, e.g. crossing without intersecting:



Generalising real analysis

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Specifically

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- some classical theorems do not generalise to $\mathbb{R}_\kappa$ e.g. adding κ-continuous need not be κ-continuous
- yet other classical theorems generalise depending on the cardinal, κ e.g. κ has tree property iff sharp functions have extreme points

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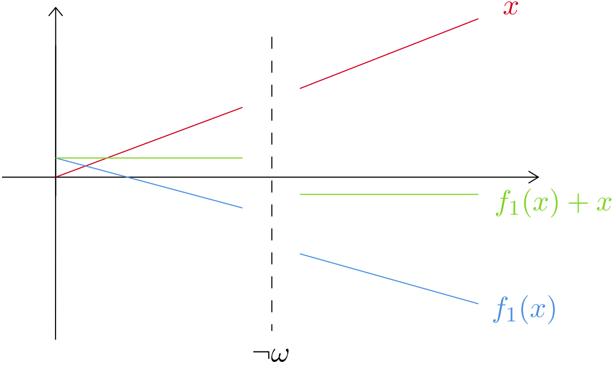

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  - the Extreme Value Theorem (continuous functions reach their extrema) generalises to $\mathbb{R}_\kappa$ iff κ has the tree property (for a generalisation of continuity, called *sharpness*)

Thank you!

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