(1) (von Neumann Hierarchy) Using recursion, formally define the $V_\alpha$ such that: $V_0 = \emptyset, V_{\alpha+1} = \mathcal{P}(V_\alpha), V_\lambda = \bigcup_{\alpha \in \lambda} V_\alpha$.

(2) Prove that:
   (a) Every $x \in V_\omega$ is finite
   (b) $V_\omega$ is transitive
   (c) $V_\omega$ is inductive
   (d*) For which $\alpha$ is $V_\alpha$ transitive, and inductive?

(3) (HF) The elements of $V_\omega$ are called \textit{hereditarily finite}. Prove that, for $x, y \in V_\omega$:
   (a) $\{x, y\} \in V_\omega$
   (b) $\bigcup x \in V_\omega$
   (c) $\mathcal{P}(x) \in V_\omega$
   (d) if $f$ is such that $\text{dom}(f) = x$ and $f(a) \in V_\omega$ for all $a \in x$ then $f[x] \in V_\omega$
   (e) If $Z \subset V_\omega$ is finite, then $Z \in V_\omega$

(4) (Independence of Replacement)
   (a) Prove that $\alpha \notin V_\alpha$ for all ordinals $\alpha$.
   (b) Using recursion, define the function $f : \omega \to \omega + \omega$ such that $f(n) = \omega + n$.
   (c) Prove that $f|_N \in V_{\omega+\omega}$ for all $N \in \mathbb{N}$.
   (d) As in (3), observe that $V_{\omega+\omega}$ is closed under Pairs, Unions, Powerset, and even Separation, and that it contains $\emptyset$ and $\omega$
   (e) What would $V_{\omega+\omega}$ contain if Replacement held “for $V_{\omega+\omega}$” (i.e. if $V_{\omega+\omega}$ is closed under Replacement)? Prove that this is a contradiction.

(5) (*for those interested in Replacement and $\mathbb{R}$) Investigate what it means for $S \subset \mathbb{R}$ to be Borel, and what it means for $S$ to be determined. Note that open and closed $S$ are determined in $Z$ (i.e. without Replacement). Meanwhile, note that proving that Borel set are determined relies essentially on Replacement.