Today we will discuss well-foundedness, its motivation, and some intuitions. If you want to be getting on with some questions, some can be found below. But today’s focus will be on understanding, not presenting solutions.

Recall that on a relation on a set, \((A, R)\), \(a \in X \subseteq A\) is the \(R\)-minimal element of \(X\) if there is no \(x \in X\) such that \(xRa\). \(R\) is well-founded on \(A\) if every \(\emptyset \neq X \subseteq A\) has an \(R\)-minimal element. The axiom of foundation (or regularity) states that for all \(X \neq \emptyset\), there is \(x \in X\) such that \(x \cap X = \emptyset\).

(1) For \(X \subseteq A\) and \((A, R)\), \(a \in X\) is the \(R\)-least element of \(X\) if \(a\) is an \(R\)-minimal element of \(X\) and \(aRb\) holds for all \(b \in X \setminus \{a\}\). Prove that if every \(A \neq \emptyset\) has an \(R\)-least element, then \((A, R)\) is a well-order.

(2) Let \((A, R)\) be well-founded. Prove that there is a unique function \(\rho\) defined on \(A\) with ordinal values such that for all \(x \in A\), \(\rho(x) = \sup\{\rho(y) + 1|xRy}\). We call \(\rho(x)\) the \(R\)-rank of \(x\).

(3) (a) Let \(A = \alpha\) be an ordinal and \(R = \epsilon_{\alpha}\). Prove that \(\rho(x) = x\) for all \(x \in A\).
(b) Let \(A = V_\omega\), \(R = \epsilon_A\) Prove that \(\rho(x) = \) the least \(n\) such that \(x \in V_{n+1}\), i.e. \(V_n = \{x \in V_\omega : \rho(x) < n\}\).
(c) Comment on what happens for \(V_\alpha\), where \(\alpha = \omega\).

(4) Prove that, assuming DC, a relation is well-founded iff it contains no countable infinite descending chains.